Preprint of a paper forthcoming in 'Mind'

The Logic of Dynamical Systems is Relevant

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Abstract

Lots of things are usefully modelled in science as *dynamical systems*: growing populations, flocking birds, engineering apparatus, cognitive agents, distant galaxies, Turing machines, neural networks. We argue that relevant logic is ideal for reasoning about dynamical systems, including interactions with the system through perturbations. Thus, dynamical systems provide a new applied interpretation of the abstract Routley-Meyer semantics for relevant logic: the worlds in the model are the states of the system, while the (in)famous ternary relation is a combination of perturbation and evolution in the system. Conversely, the logic of the relevant conditional provides sound and complete laws of dynamical systems.

Keywords: Dynamical systems, Relevant logic, Perturbation, Relevant conditional, Routley-Meyer semantics

1 Introduction

To really understand the workings of a system (be it physical, biological, social, economical, computational), we shouldn't limit ourselves to observing it: we should also interact with it. We may want to know the effects of administering a certain medication; understand the consequences of implementing a certain tax policy in a society; test how a new material reacts to certain chemicals; check whether an artificial neural network behaves as intended on slightly different inputs. In each case, we wonder about the truth value of conditionals of the form:

1. Whenever we perturb the system from its current state into a state where φ, it will then evolve into a state where ψ.

Such *perturbation conditionals*, when true, can be regarded as *laws* describing the behaviour of the system: that φ is followed by ψ. Knowing such laws matters for the explanation and interpretation of the system, for the scientific goal of predicting how the system may behave, but also for the technological goal of verifying that the system robustly behaves as we want it to.

Causal models provide a well-known approach to perturbation conditionals (e.g. Hitchcock 2023; Pearl 2009): systems are represented as so-called structural causal models. Perturbations are formalized as interventions changing such models. Perturbation conditionals are then taken as structural counterfactuals: 'If φ were made true by an intervention, then ψ would be true'. This gives a semantics to perturbation conditionals; there is quite some discussion of their logic (e.g. Briggs 2012; Galles and Pearl 1998; Halpern 2000; Ibeling and Icard 2020; Zhang 2013). We will pursue a different and novel approach to perturbation conditionals, giving a logic and a semantics for them: we will analyze systems as *dynamical systems*, and their logic will turn out to be *relevant logic*. We don't oppose the causal model approach, rather develop a different and hopefully enlightening perspective on perturbation conditionals.

Our own view gets off the ground by observing that we can rephrase the perturbation conditional 1 as:

2. For all states y and z of the system, if there is a perturbation moving the system from its current state x to y from which it evolves to z, then if φ holds at y, then ψ holds at z.

This says that a certain conditional—let's write it '$\varphi \rightsquigarrow \psi$'—holds at the current state x of the system. Let's write this as '$x \vDash \varphi \rightsquigarrow \psi$'. Then we take a ternary relation on the states of the system, $Rxyz$: 'A perturbation changes the system from state x to y, from which it evolves to z'. With this notation, we can further rephrase 1—i.e., $x \vDash \varphi \rightsquigarrow \psi$—as

3. For all states y and z with $Rxyz$, if $y \vDash \varphi$, then $z \vDash \psi$.

Now these are precisely the truth conditions for the conditional in relevant logic, under its widely discussed—and (in)famously abstract—Routley-Meyer semantics (R. Routley 1979; R. Routley and R. Meyer 1972a,b,

1. $\varphi \to (\psi \to \varphi)$
2. $\neg\varphi \to (\varphi \to \psi)$
3. $\varphi \to (\psi \to \psi)$
4. $\varphi \to (\psi \vee \neg\psi)$
5. $(\varphi \wedge \neg\varphi) \to \psi$

Figure 1: The paradoxes of the material and strict conditional

1973). We will work out this key idea as a precise semantics of perturbation conditionals, whose sound and complete logic is axiomatized by relevant logic.

That a non-classical logic—one postulating situations where the law of self-implication $\varphi \to \varphi$ can fail!—should be the logic of dynamical systems is explained by our interpretation of the conditional in terms of perturbation and evolution. It might well be that we perturb a state x to a state where φ is true (say, mitigating pain by taking painkillers), but then the system evolves into a state where φ is not true anymore (the pain coming back when the painkillers wear off), so x doesn't make true $\varphi \rightsquigarrow \varphi$. Besides addressing perturbation conditionals, this will also deal with a long-standing open problem: providing an applied interpretation of the abstract Routley-Meyer semantics. Relevant logic turns out to be the logic of dynamical systems, and the logic of relevant conditionals turns out to be the logic of the laws governing dynamical systems.

Experts of either dynamical systems or relevant logic may not know much about the other topic; so we provide very short introductions to both. We first briefly outline relevant logic (section 2) and its Routley-Meyer semantics (section 3). Section 4 makes sense of its treatment of negation and section 5 sketches the struggles so far in interpreting its abstract ternary relation. We then turn to dynamical systems: In section 6, we give an informal introduction, which we formalize (including a notion of perturbation) in section 7. Section 8 shows how this gives rise to a Routley-Meyer model and thus gives a semantics for a logic of dynamical systems including perturbation conditionals. Section 9 shows that, conversely, every Routley-Meyer model arises, up to equivalence, from a dynamical system. This gives the desired soundness and completeness results. Section 10 then investigates this bridge between systems and logics by starting to explore how certain subclasses of systems are characterized by certain logical axioms. We conclude in section 11.

2 Relevant logic

Relevant logic was to capture a notion of conditionality free from the paradoxes of the material and strict conditional listed in figure 1. These offend our sense that a conditional should only hold when its antecedent has some connection to its consequent; otherwise, whatever grounds the truth of the former cannot be transmitted to the latter. 'If snow is white, then if the moon is made of green cheese, then snow is white' (an instance of 1) seems to make snow's whiteness depend on a silly falsehood, given only that snow is white; 'If Aidan doesn't smoke, then if he does the Earth will implode' (an instance of 2) seems to make the end of life on Earth depend on Aidan's smoking, given only that he doesn't. Even if one interprets the conditional as strict or modally qualified, this is not enough for 3–5: 'If Midori is happy, then if grass is green, then grass is green' (an instance of 3) seems to relate the fate of self-implication to Midori's mood. 'If I'm a monkey's uncle, then either 2 is prime or it's composite' seems to make a necessary truth depend on one's having primates of other species as relatives. However, in normal modal logic, the consequents of 3 and 4 fail in no possible scenario.

Anderson and Belnap (1975) held as a necessary condition for a conditional $\varphi \to \psi$ to be valid, or a theorem, that φ and ψ share some sentential variable, thus capturing syntactically the idea of a connection between antecedent and consequent. This was called the Variable Sharing Property (VSP) (Dunn and Restall 2002, p. 27). Anderson and Belnap first came up with proof-theoretic logical systems ensuring that no conditional would count as a theorem unless it had the VSP.[1] Take a sentential language with a countable set At of atoms $p_1, p_2, \ldots$, negation $\neg$, conjunction $\wedge$, disjunction $\vee$, conditional $\to$. We use $\varphi, \psi, \chi, \theta, \varphi_1, \varphi_2, \ldots$ as meta-variables for formulas. The well-formed formulas are the atoms and, if φ and ψ are well-formed, so are $\neg\varphi, (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \to \psi)$. (We will normally omit the outermost parentheses; a biconditional $\leftrightarrow$ can be defined out of $\to$ and $\wedge$ the usual way.) The axioms and rules of the the *basic positive* relevant logic B^+ are given in figure 2. One can add to the positive logic principles for negation such as:

(A7) $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$
(A8) $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$
(A9) $\varphi \leftrightarrow \neg\neg\varphi$
(R4) $\varphi \to \psi \vdash \neg\psi \to \neg\varphi$.

[1] Besides the one of 'content connection', there's another informal idea at the origins of the Anderson-Belnap relevant tradition: that of 'making real use' of assumptions. What's bad, e.g., with paradox 1, is that if φ is already given, ψ is really doing nothing to get us to infer φ. We don't much talk of this other informal idea as it becomes showy especially if one sees relevance proof-theoretically (compare Dunn and Restall 2002), whereas we have a models-first approach.

The axioms

(A1) $\varphi \to \varphi$
(A2) $\varphi \to (\varphi \vee \psi)$ and $\psi \to (\varphi \vee \psi)$
(A3) $(\varphi \wedge \psi) \to \varphi$ and $(\varphi \wedge \psi) \to \psi$
(A4) $(\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee \chi)$
(A5) $((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$
(A6) $((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$

The rules

(R1) $\varphi, \varphi \to \psi \vdash \psi$
(R2) $\varphi, \psi \vdash \varphi \wedge \psi$
(R3) $\varphi \to \psi, \chi \to \theta \vdash (\psi \to \chi) \to (\varphi \to \theta)$

plus their disjunctive forms (the disjunctive form of a rule $\varphi_1, \ldots, \varphi_n \vdash \psi$ is $\chi \vee \varphi_1, \ldots, \chi \vee \varphi_n \vdash \chi \vee \psi$).

Figure 2: The relevant logic B^+

The following extensions of B^+ are often considered:

$$\mathsf{BM} = \mathsf{B}^+ + \text{(A7–8)} + \text{(R4)} \qquad\qquad \mathsf{B} = \mathsf{BM} + \text{(A9)}.$$

System B was taken as the *basic* relevant logic. There are stronger relevant systems; one of Anderson and Belnap's favourite ones is called R. We'll talk about that in section 10. Until then, we will be concerned with the basic (positive) relevant logic.

3 The Routley-Meyer Semantics

The issue of finding a semantics entered the relevantist agenda early on:

> Yea, every year or so Anderson & Belnap turned out a new logic, and they did call it E, or R, or E_I, or $P - W$, and they beheld such logic, and they were called relevant. And these logics were looked upon with favor by many, for they captureth the intuitions, but by many they were scorned, in that they hadeth no semantics. Word that Anderson & Belnap had made a logic without semantics leaked out. Some thought it wondrous and rejoiced, that the One True Logic should make its appearance among us in the Form of Pure Syntax, unencumbered by all that set-theoretical garbage. Others said that relevant logics were Mere Syntax. (R. Routley and R. Meyer 1973, p. 194)

Initial algebraic semantics based on De Morgan lattices didn't seem too enlightening: they looked like *pure*, not *applied* semantics (Carnap (1948), Dummett (1973), and Plantinga (1974)). Pure semantics consists of mathematical structures that interpret the formal language but are themselves uninterpreted. Applied semantics concerns the interpretation of the mathematical structures as representing something we already have some independent grasp of.

The frame semantics developed by R. Routley and V. Routley (1972) and R. Routley and R. Meyer (1972a,b, 1973) seemed similar enough to Kripke or possible worlds semantics for modal logic to warrant optimism on a plausible interpretation. First let's see what a core issue was: paradoxes like 3–5 in figure 1 are conditionals featuring (classical and normal modal) logical truths in their consequent or falsities in their antecedent. These are supposed to hold everywhere and, respectively, nowhere in logical space. Then where can we find situations falsifying the former, or verifying the latter, while at the same time retaining their status as logical truths/falsities? Routley and Meyer addressed the issue by resorting to situations different from classically possible worlds. To see how these work, we will introduce (a small variation on) the simplified semantics for relevant logics due to Priest and Sylvan (1992) and Restall (1993); this is easier to work with and has become somewhat canonical after being adopted in Priest (2008)'s classic textbook.

A *relevant model* for the language above is a tuple $M = (W, 0, R, C, i)$ where:

- W is a set of worlds
- $0 \in W$ is the base world
- $R \subseteq W^3$ is a ternary relation
- $C \subseteq W^2$ is a binary relation
- $i : W \times \mathsf{At} \to \{1, 0\}$ is an interpretation function

such that, for all $x, y \in W$:

$$R0xy \text{ if and only if } x = y \qquad (1)$$

This is called the *normality condition*; whether in the original Routley-Meyer semantics or in the Priest-Sylvan-Restall simplified variant, it makes for a standard way of marking a difference between the normal or base world (or, worlds) and other worlds in the models (see Priest 2008: 189-90 for discussion). The truth clauses go as follows. (In the metalanguage we use $x, y, z, x_1, x_2, \ldots$, ranging over worlds; $\Rightarrow, \Leftrightarrow, \&, \forall, \exists$, with the usual reading; and $\nVdash$ for 'not $\Vdash$'.)

(At) $M, x \Vdash p \Leftrightarrow i(x, p) = 1$

$(\neg)$ $M, x \Vdash \neg\varphi \Leftrightarrow \forall y \in W(xCy \Rightarrow y \nVdash \varphi)$
$(\wedge)$ $M, x \Vdash \varphi \wedge \psi \Leftrightarrow M, x \Vdash \varphi \;\&\; M, x \Vdash \psi$
$(\vee)$ $M, x \Vdash \varphi \vee \psi \Leftrightarrow M, x \Vdash \varphi \text{ or } M, x \Vdash \psi$
$(\rightarrow)$ $M, x \Vdash \varphi \rightarrow \psi \Leftrightarrow \forall y, z \in W(Rxyz \;\&\; y \Vdash \varphi \Rightarrow z \Vdash \psi)$

We'll omit reference to M when this is clear from context. Logical consequence is truth preservation at the base world in all models; with Σ a set of formulas:

$$\Sigma \vDash \varphi \Leftrightarrow \text{for all } M\text{: } M, 0 \Vdash \psi \text{ for all } \psi \in \Sigma \Rightarrow M, 0 \Vdash \varphi.$$ [2]

Logical truth, $\vDash \varphi$, is entailment by the empty set: truth at the base world in all models.

We'll call *positive* the models where we take away C and forget about the truth conditions for negation. The logic B^+ is (strongly) sound and complete with respect to the class of positive models (Priest and Sylvan 1992). We'll call *ordered* the models $M = (W, 0, R, C, \leq, i)$ with a partial ordering $\leq$ on W which is *hereditary* or *preservation*: if $x \Vdash \varphi$ and $x \leq y$, then $y \Vdash \varphi$.[3] One can then regard unordered models as those where the order is the identity relation (known as discrete order).

We called the points in W 'worlds', but they're no classically possible worlds. Let's check how the semantics invalidates paradoxes 3–5 of figure 1. Consider the model $M = (W, 0, R, C, i)$ consisting of four distinct worlds, say $W = \{0, a, b, c\}$, a relation R where only $R0ww$ (for $w \in W$) and $Rabc$ hold, a compatibility relation with only aCb, and an interpretation where only p is true at a and q at b. Then

- $0 \nVdash p \rightarrow (q \rightarrow q)$ because $a \Vdash p$ but $a \nVdash q \rightarrow q$ (since $Rabc$, $b \Vdash q$, but $c \nVdash q$),
- $0 \nVdash p \rightarrow (q \vee \neg q)$ because $a \Vdash p$ but $a \nVdash q \vee \neg q$ (since $a \nVdash q$ and $a \nVdash \neg q$ for aCb and $b \Vdash q$)

[2]This is the textbook definition (e.g. Priest 2008, p. 10.2.6). Many thanks to an anonymous referee for mentioning a discussion of Anderson, Belnap, and Dunn (1992, p. 196): They call this definition the 'official' consequence to stress that it, itself, is not a relevant consequence: the $\Rightarrow$ is not a relevant conditional in the metalanguage, but the usual notion of entailment in our classical metalanguage. This definition affords the strong version of soundness and completeness (see the next paragraph) and is in line with the common convention of using a classical metalanguage when discussing a non-classical logical system. It seems particularly suited in our case where we claim that dynamical systems—standard objects in classical mathematics—provide an interpretation to the specific logical system of relevant logic.

[3]Sometimes they are defined differently (e.g. Restall 1993, p. 498): $\leq$ is a binary relation (*containment*), required to satisfy some easy-to-check properties that precisely ensure the desired heredity condition.

- $0 \not\Vdash (p \wedge \neg p) \rightarrow q$ because $a \Vdash p \wedge \neg p$ (since $a \Vdash p$ and $a \Vdash \neg p$ for if aCw, then $w = b$, and $b \not\Vdash p$) but $a \not\Vdash q$.

The semantics validates $\varphi \rightarrow \varphi$: it is true at the base world 0 in any model M, because if $R0xy$ and $x \Vdash \varphi$, then also $y \Vdash \varphi$ since $x = y$. Self-implication can only fail at non-base worlds. (Flag the point: the base world is *special* in this respect – and rightly so: we'll get back to this.)

Thus, 3–5 are invalidated thanks to points of evaluation which can fail self-implication, can be locally inconsistent (making true both a formula and its negation), and can be incomplete (making true neither). So they cannot represent classically possible worlds. What things *can* they represent, then, such that the relations R and C between such things, which give the truth conditions for the conditional and negation, make sense? That's what an applied semantics has to answer.

4 Negation and (In)Compatibility

For C, we don't even need to be specific on the nature of the things. That's because there's a tradition, dating back at least to the Birkhoff–von Neumann–Goldblatt characterization of ortho-negation in quantum logic (Birkhoff and von Neumann 1936; Goldblatt 1974), and developed by Berto (2015), Berto and Restall (2018), Došen (1986), Dunn (1993), and Restall (1999), and many others, accounting for the meaning of negation via the fundamental notions of *compatibility* and its polar opposite, *incompatibility* or *exclusion*. And incompatibility is so basic to (our experience of) the world (Kinkaid 2020), that it's easy to make sense of it as holding between the most diverse kinds of things.[4] So just take 'xCy' as saying that x and y are compatible. Then $(\neg)$ has it that $\neg\varphi$ holds at a point iff φ fails at all compatible points. Using incompatibility I, i.e., the complement of compatibility C, one could equivalently go for:

$(\neg)$ $M, x \Vdash \neg\varphi \Leftrightarrow \forall y \in W(y \Vdash \varphi \Rightarrow xIy)$

[4]E.g., one could see it as a relation between features of objects: *being square* rules out *being circular*; *being prime* rules out *being composite*; *being entirely located here* rules out *being entirely located over there* at the same time. Or, it could hold between states of affairs: the ball's being red all over rules out its being blue all over; this number's being prime rules out its being composite; the table's being wholly in the garden rules out its being wholly in the kitchen. Or, it could hold between the corresponding propositions: that one is in the garden rules out that one is in the kitchen; etc. Or, it could hold between pieces of information, or of evidence, or whatnot. See Berto 2015, whose metaphysical story is summarized in this footnote.

So $\neg\varphi$ holds at a point iff any point where φ holds is incompatible with it. Points in W making true formulas, then, could be (in)compatible due to properties of the objects that exist at them; or, to states of affairs that obtain at them, or, to the states of affairs they themselves are; or, to propositions they support, or include; or, to pieces of information or evidence they convey; or whatnot. The setting makes plain intuitive sense of the meaning of negation, because we utter negations to exclude things or express incompatibilities (Kinkaid 2020; Mares 2004; Price 1990; Restall 1993). By imposing conditions on C, one can validate various principles involving negation. Here are three:

- $\forall xy(xCy \Rightarrow yCx)$ (Symmetry)
- $\forall x\exists y(xCy)$ (Seriality)
- $\forall x(\exists y(xCy) \Rightarrow \exists z(xCz \,\&\, \forall w(xCw \Rightarrow w \leq z))$ (Convergence)

It seems natural that (in)compatibility should be serial and symmetric (for some reasons why, see Restall (1999), Berto (2015)); we'll get back to Symmetry when we discuss our interpretation via dynamical systems, though). What convergence essentially does is guaranteeing that each world has a maximally compatible mate if it is compatible with anything at all. If one buys these conditions, each $x \in W$ will have a maximal compatible mate x^*, and $(\neg)$ becomes equivalent to

$(\neg^*)$ $M, x \Vdash \neg\varphi \Leftrightarrow M, x^* \nVdash \varphi$.

Say a *star-ordered* model is an ordered model satisfying Symmetry, Seriality, and Convergence. One can also take the star operation as primitive: a *star model*, then, is $M = (W, 0, R, *, i)$, with the truth conditions for negation given directly as per $(\neg^*)$. The above story, however, shows how the star operation arises naturally from compatibility and order: see again Restall (1999).

The relevant logic BM is sound and complete with respect to star models (Priest and Sylvan 1992). If we additionally impose the condition $x = x^{**}$, we get *Routley star* models, with respect to which the logic B is sound and complete (Priest and Sylvan 1992). Such an involutive (or period-two) star operation was used in the original (R. Routley and R. Meyer 1972a; R. Routley and V. Routley 1972) to give the semantics for negation—called the *Routley star*.

One may not like the idea that the base world, where logical truth and validity are recorded, could be inconsistent or incomplete. In the Routley star setting, one can accommodate the worry by stipulating that $0^* = 0$, which ensures that negation behaves classically at 0: exactly one of φ and $\neg\varphi$ will be true there, for all φ. This gives a logic stronger than B, but all the counterexamples to the paradoxes still go through.

Making sense of the ternary R is way more work.

5 The Ternary Relation

A number of interpretations understand the points in W as states of or conduits for information, and the ternary relation in terms of information transmission (Beall et al. 2012; Golan 2023; Mares 2004; Restall 1995; Tedder 2021; Urquhart 1972). Both Restall and Mares took the points to represent situations in the sense of Barwise and Perry (1983)'s situation semantics. These are information-supporting structures that need not be maximal, as they can fail to support either positive or negative information about certain topics. The current meteorological situation in Oxford does not support the information that it's raining in New Jersey, nor the one that it isn't raining there. Situations can also be taken as abstract objects representing logical impossibilities and have been developed by Barwise and Seligman (1997) into a general theory of information flow in distributed systems. The partial ordering in enriched models is then understood as information-inclusion: '$x \leq y$' means that all the information supported by x is also supported by y. One may also have an (idempotent, commutative, associative) operation of fusion, $\oplus$, the pooling together of pieces of information, and define information-inclusion out of it, the usual way, as $x \leq y =_{df} x \oplus y = y$.

Then one can read '$Rxyz$' as saying that x is a situation that acts as a conduit of information, allowing it to be transmitted from situation y to situation z. This makes sense of the truth conditions for the relevant conditional, as per ($\rightarrow$): when x allows the information that $\varphi \rightarrow \psi$ to flow from y to z, and y supports the information that φ, then z should support the information that ψ.

Here is one thing '$Rxyz$' *cannot* mean in the information-theoretic reading (Dunn and Restall 2002; Priest 2008): it cannot just mean that z is or includes the information obtained by pooling together x and y, i.e., $x \oplus y \leq z$. This would make $\psi \rightarrow \psi$ true at all points x (if $Rxyz$ and $y \Vdash \psi$, then $y \leq x \oplus y \leq z$, so, by preservation, also $z \Vdash \psi$), hence paradox 3 is validated. This complicates the informational interpretation of R: 'x allows information to flow from y to z' cannot be understood in the plain mereological terms that fusing y to x yields a result in z, since *preservation* must fail: $Rxyz$ cannot imply that y is informationally contained in z. As Priest (2008, p. 207) has it: 'The problem now is to make sense of the metaphor of information flow—hardly a transparent one.'

To show how R makes sense in *our* setting in spite of failures of preservation, let us now introduce dynamical systems.[5]

[5]For an early version of this idea, see Hornischer (2021, sec. 3.6.2).

6 Dynamical Systems, Informally

A dynamical system consists of a *state space* and a *dynamics* on it. The former is the collection of states that the system can be in; the latter represents how the system evolves from one state to another (List and Pivato 2021). Lots of things can then be seen as a dynamical system: gases in motion, agents in financial markets, computers. Let's consider a standard textbook example (e.g., Strogatz 2015, sec. 2.3/10.2; May 1976).

Example a (Running example: population growth)**.** A population of insects breed seasonally without overlapping generations. A state of this system is described by the number of insects that the system has in this state. It will be convenient to describe the state by the percentage of the maximal population: a state is given by a number x between 0 and 1. We want to understand how this evolves over time, i.e., over the seasons. When there are few insects (x is close to 0), there will be more in the next generation: the available resources plentifully support the few. However, when there are many insects (x is close to 1), there will be fewer in the next generation: the available resources don't support the many. To model this, we look for a function T that, when given the current state x as input, describes the population $T(x)$ in the next season. We will do this in the next section.

Dynamical systems can have a discrete or continuous state space. Our population example with absolute numbers as states would be discrete, but with the percentages as states it is continuous. Systems can be time-discrete, when they develop in discrete time steps, or time-continuous, when subject to continuous change. Our population example is time-discrete. And systems can be deterministic, when each state has a unique successor; non-deterministic or stochastic, when that's not the case. The implicit assumption in modeling our population dynamics as a function is that it is deterministic: for every current state x, there is a unique next state $T(x)$.

Let's end this section with three more examples.

First, a gas in a box can be a time- and state-continuous deterministic dynamical system: a state of the system is given by the position and momentum of each gas molecule in the box. The dynamics is given by the laws of motion of classical mechanics. This type of example is extremely general: differential equations are the language of physics, chemistry, engineering, and many other sciences, and any solution to a differential equation yields such a dynamical system (for details, see Teschl 2012, sec. 6.2).

Second, Turing machines are time- and state-discrete dynamical systems: a state of a Turing machine at a time is given by what's written on its tape, the part of the tape the machine is scanning at the time, and its internal state. The dynamics is given by the program of the machine, fixing the next state (or one of the possible next states, if it's non-deterministic) given what the machine is reading at the current state and its current internal state.

Third, training artificial neural networks is a time-discrete, state-continuous dynamical system. A state describes the weights of all the connections in the network. And the new set of weights is computed, for example, from a batch of data points: the network's output on the data is compared to what should be the correct output according to the data; then the weights are adjusted (using the backpropagation algorithm) so as to provide outputs closer to the correct ones. Since the batch is sampled randomly, the dynamics is non-deterministic.

7 Dynamical Systems, Formally

There are many formal notions of a deterministic system, differing in mathematical structure and dynamics. They all are so-called (left) actions α of a monoid $(M, +, 0)$ on a set X. Here X describes the state space of the system, the monoid describes the notion of time, and the function $\alpha : M \times X \to X$ describes the dynamics: 'If the system is in state $x \in X$ now, then after time $m \in M$, the system is in state $\alpha(m, x)$' (so one requires $\alpha(0, x) = x$ and $\alpha(m + n, x) = \alpha(m, \alpha(n, x))$). A time-continuous system would use as monoid the real numbers $(\mathbb{R}, +, 0)$, while a time-discrete system would use the natural numbers $(\mathbb{N}, +, 0)$. In the time-discrete case, the dynamics is already described by the '1-step dynamics' $T = \alpha(1, \cdot) : X \to X$ mapping each state to the next state, because

$$\alpha(0, x) = x, \qquad \alpha(1, x) = T(x), \qquad \alpha(2, x) = T(T(x)), \qquad \ldots.$$

If the state space X is continuous, one formalizes this by assuming it to be a topological space or a probability space and by requiring the dynamics to preserve this additional structure (i.e., being continuous or measure-preserving, respectively).

Among the main classes of systems that dynamical systems theory studies are the so-called topological systems. These are state-continuous and time-discrete deterministic systems. Historically they became important through the work of Poincaré because they allowed for the analysis of solutions to differential equations also in absence of *explicit* solutions (which is most often the case). The formal definition is taken from the excellent textbook by de Vries (2014, p. 1). The precise definitions of mathematical terms are added as footnotes, but an intuitive understanding should be enough for us.

Definition 1. A *topological system* is a pair (X, T) where X is a Hausdorff topological space and $T : X \to X$ is continuous.[6] Sometimes it is additionally assumed that X is compact and that f is bijective—if this is the case, we call (X, T) a *standard* topological system.

Here is what this formal definition looks like for our running example.

Example b (Example a continued)**.** Our example system is a population of insects. Its states are given by numbers x between 0 and 1 describing the percentage of the maximal population. So the state space is the unit interval $X := [0, 1]$ with its usual topology. For the function $T : X \to X$ describing the dynamics we required an increase when x is small and a decrease when x is large. Famously, the *logistic map* does this:

$$T(x) := 2x(1-x).$$[7]

This is visualized in figure 3: For small x (with $x < 0.5$), the population will increase ($T(x)$ is above the dotted line), unless there weren't any insects to start with (i.e., $x = 0$, in which case $T(x) = 0$). For large x (with $x > 0.5$), the population will decrease ($T(x)$ is below the dotted line). For $x = 0.5$, the population remains stable: $T(x) = x$ (the population is in equilibrium). For the extreme points $x = 0$ (no population), we also have $T(x) = x$; and for $x = 1$ (maximal population), we have $T(x) = 0$ (no population). So $x = 0$ and $x = 0.5$ are the only fixed points (i.e., points x with $T(x) = x$). But only $x = 0.5$ is attracting: any $0 < x < 1$ converges under T to 0.5. Thus, we have a clear idea of the long-term dynamics.

To explicate when a perturbation conditional is true at a state, we need to make some more structure explicit: (1) what it means to perturb the system, and (2) what it means that a property holds at a state or doesn't hold (i.e., the state is incompatible with it). We call the resulting structure an *interactive topological system*:

Definition 2. An *interactive topological system* S is a tuple (X, x_0, T, A, P, C, i) where:

[6] A *topological space* is a set X together with a set τ of subsets of X that contains X and the empty set $\emptyset$ and is closed under finite intersection and arbitrary union. The elements of X are called *points* and the elements of τ are called *open sets*. Complements of open sets are called *closed*. A topological space is *Hausdorff* if, for any two distinct points x and y in X, there are disjoint open sets U and V with $x \in U$ and $y \in V$. We often only refer to the space by X and take τ to be given by context. A function $f : X \to Y$ between two topological spaces is *continuous* if for any open set V of Y, the preimage $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is an open set of X.

[7] Instead of 2, we could choose another constant $0 \leq r \leq 4$. For $r \leq 2$, the dynamics is quite stable (which we choose here to keep things simple), but for values $r > 2$ it can become mesmerizingly complex (May 1976).

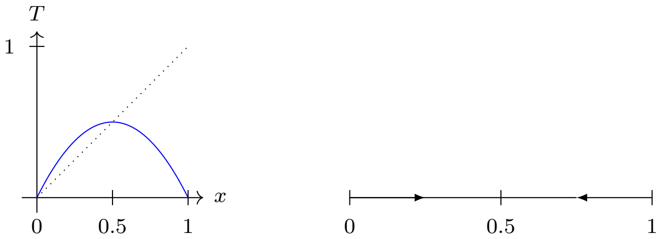

Figure 3: The dynamics of the logistic map on the unit interval.

- X is a Hausdorff topological space: the *state space*.
- $x_0 \in X$ is a state: the *initial state*.
- $T : X \to X$ is a continuous function: the *dynamics*.
- A is a topological space: the *space of perturbations*.
- $P : X \times A \rightrightarrows X$ is a multifunction (i.e., a function that maps each element of its domain $X \times A$ to a subset of its codomain X): the *perturbation function*. We require F to be 'appropriately continuous' which here means closed-valued, upper hemicontinuous, and with closed domain.[8]
- $C \subseteq X^2$ is a binary relation: the *compatibility relation*.
- $i : X \times \mathsf{At} \to \{0, 1\}$ is a function: the *interpretation function*.

We call S *standard* if X is a compact Hausdorff space and T is bijective (hence a homeomorphism). We call S *simplified* if $A = X$. A *positive* interactive topological system is defined in the same way but without the compatibility relation.

Let's explain the parts of the definition in turn and then consider it for our running example. The part (X, T) just describes the underlying topological system. Distinguishing a state $x_0 \in X$ is typical in dynamical systems theory: it fixes the current state of the system, starting from which it is analyzed (e.g., the initial condition in an initial value problem whose solution is the dynamical system).

The interpretation function i allows us to speak about properties of the system: In our insect population system, consider p representing 'The population is highly, but not extremely populated'. So p is true at a state

[8]Recall from set-valued analysis (Aliprantis and Border 2006, ch. 17) that a multifunction $F : X \rightrightarrows Y$ on topological spaces X and Y is *closed-valued* if each $F(x)$ is a closed subset of Y, and F has a *closed domain* if $\{x \in X : F(x) \neq \emptyset\}$ is a closed subset of X. Finally, F is upper hemicontinuous if, for all $x \in X$ and open $V \subseteq Y$ with $F(x) \subseteq V$, there is an open $U \subseteq X$ with $x \in U$ and, for all $x' \in U$, we have $F(x') \subseteq V$. To apply this to the perturbation function P, we take the product topology on $X \times A$ (the least topology where all $U \times V$ are open for $U \subseteq X$ open and $V \subseteq A$ open).

x according to i (i.e., $i(x,p) = 1$) iff, say, $0.7 < x < 0.9$, i.e., the insect population is higher than 70% of the maximal population but below 90%.

The compatibility relation is to express what it means that the system currently does *not* have property p. Following section 4: not having p means not being compatible with any state having p. If we take C to be the identity relation, we can recover the classical reading where state x not having property p means $i(x,p) = 0$. But, for a more constructive reading, we can also take C to be indistinguishability by the available measurements. Then state x does not having property p iff we can conclusively show that the system currently does not have p, because p fails in all the states that, based on our observations, the system could be in. For example, in our insect population, we can plausibly only measure the number of insects up to a 5% error of measurement, so x does not having property p iff for no ϵ with $-0.05 \leq \epsilon \leq +0.05$, we have $0.7 < x + \epsilon < 0.9$ (i.e., $x \leq 0.65$ or $x \geq 0.95$). There are yet further options: e.g., x is compatible with y if y can be obtained from x with a small perturbation, which need not even be reflexive or symmetric. So generally we just assume C to be a binary relation.

Finally, the perturbation function says: if the system is in state $x \in X$ and perturbation $a \in A$ acts on the system from the outside, it will perturb the system into a state in $P(x,a)$. In our running example, a could be the action of adding one million insects which, say, is 10% of the maximal population. The reason why we don't assume P to be a function but only a multifunction is that perturbations usually are not infinitely precise: if we apply perturbation a to the system in state x, we usually cannot guarantee a unique resulting state $P(x,a)$ but only a 'ballpark' of states $P(x,a) \subseteq X$. In the example, it wouldn't be feasible to count that it was precisely 10% of the maximal population that we added, it could also have been 1% more or less, so $P(x,a) = \{y \in [0,1] : x + 0.09 \leq y \leq x + 0.11\}$.

We should expect the perturbations to interact with the spatial structure of the system: If we change the current state x and/or perturbation a by a bit, then the resulting states $P(x,a)$ should also only change a bit. If P was a function, this would precisely be achieved by requiring P to be continuous. But since P is a multifunction, the next best thing is to assume it to be appropriately continuous in the technical sense above.[9]

[9] Are perturbations just *any* kind of state-change, a helpful referee asks? We think it's not up to the formal semantics to further constrain what counts as a perturbation. Even a fixed dynamical system will be amenable to very different sorts of perturbation (see, e.g., example d vs. e below). What's up to the semantics is to represent the structural properties shared by all sorts of perturbation: these are the conditions we impose on P. This is analogous to Kripke models: They represent the informal idea that y is a possibility for x as a binary relation Rxy. Again, this captures the structural properties that any instance of the informal notion shares, but it does not

Requiring the system to be simplified is motivated thus: although perturbations are not states, often we can identify the two (Leitgeb 2005). For example, the perturbation of adding 10% of the maximal population can be identified with the state $x = 0.10$.

In sum, here is our running example as an interactive topological system.

Example c (Example b continued)**.** In our insect population system, we have:

- The state space $X = [0, 1]$ of possible percentages of the maximal population.
- The initial state, say, $x_0 = 0.6$.
- The population dynamics $T : X \to X$ given by $T(x) = 2x(1 - x)$.
- The space of perturbation $A := [0, 1]$ describing how many insects we externally add to the population, measured in percent of the maximal population.
- The perturbation function $P : X \times A \rightrightarrows X$ given by

$$P(x, a) = \Big\{ y \in [0, 1] : x + a - 0.01 \leq y \leq x + a + 0.01 \Big\}.$$

- The compatibility relation xCy given by $|x - y| \leq 0.05$.
- We consider p representing 'The population is highly populated' and q representing 'The population is healthy', and no further properties. So we define the interpretation function $i : X \times \mathsf{At} \to \{0, 1\}$ as (given a state $x \in X$): $i(x, p) = 1$ iff $0.7 < x < 0.9$, and $i(x, q) = 1$ iff $0.4 < x < 0.6$; and $i(x, r) = 0$ for all other atomic sentences r.

8 Logic of dynamical systems: interpreting relevant models

We can now provide a semantics for perturbation conditionals as well as for the other connectives $\wedge, \vee, \neg$; so we can speak of a *logic of dynamical systems*.[10] The intuitive interpretation goes thus: atomic sentences represent basic properties of the system; as for the complex properties:

- The property $\varphi \wedge \psi$ holds at a state iff both the property φ and the property ψ hold at that state.
- The property $\varphi \vee \psi$ holds at a state iff either the property φ or the property ψ holds at that state.

answer which worlds exactly are possibilities for, say, our actual world.

[10] For other logics of dynamical systems, see, e.g., Leitgeb (2005), Kremer and Mints (2007), Platzer (2012), or Fernández-Duque (2012).

- The property $\neg\varphi$ holds at a state iff φ fails at all states that are compatible with that state (e.g., observationally indistinguishable).
- The property $\varphi \rightsquigarrow \psi$ holds at a state iff whenever we perturb the system from that state into a φ-state, it will evolve into a ψ-state.
- (We might add: the property $\varphi \to \psi$, which holds globally iff any φ-state also is a ψ-state.)

For our insect population, we want to know if the conditional $p \rightsquigarrow q$ is true in the current state of the population: that if we perturb the system to be highly populated, the system will recover again, i.e., evolve into a state where the population is healthy.

To get an explication of this intuitive semantics, we interpret our system as a relevant model and use its formal semantics. Naturally, the worlds of the model are the states of the system (though we add one special base world below). The interpretation function is that from the system, and world-incompatibility is state-incompatibility.[11] So atomic properties and $\wedge, \vee, \neg$ are straightforwardly interpreted. The trick for $\rightsquigarrow$ is, as mentioned in the introduction, to read '$Rxyz$' as: there is a perturbation moving the system from x into state y from which it evolves to z. Then the above intuitive meaning of $\varphi \rightsquigarrow \psi$ is just the relevant conditional:

$$x \Vdash \varphi \rightsquigarrow \psi \text{ iff } \forall y, z \in X : Rxyz \;\&\; y \Vdash \varphi \Rightarrow z \Vdash \psi.$$

This reading of R is made precise as:

$$Rxyz \Leftrightarrow \exists a \in A : y \in P(x, a) \text{ and } \lim_{n\to\infty} T^n(y) = z,$$

where the second condition means that the sequence $y, T(y), T(T(y)), \ldots$ (which is known as the *orbit* of y) converges in the space X to z.[12]

It remains to specify the base world, which is to satisfy the normality condition (1). The idea is that the base world represents the *state of*

[11] Once (in)compatibility is understood as a relation between states of dynamical systems, should we (still) accept Symmetry? Perhaps (thanks to a helpful referee for this) transforming a into b (changing hydrogen and oxygen into water) is sometimes feasible, while transforming b back into a (water into hydrogen and oxygen) is more work. One of us has argued in print (Berto 2015) that the idea that (in)compatibility may not be symmetric imports intuitions from the asymmetry of causal processes; and it's not clear that these should be embedded in a general semantics for negation. But if they should, not much hinges on that. We'll then have relevant logics where Double Negation Introduction fails (a well-known effect of the failure of Symmetry: see Restall 1999, Dunn 1993, Simonelli 2024). DNI will at most hold in a restricted class of models where compatibility is symmetric, just as the Brouwerische axiom B of normal modal logics only holds in a restricted class of Kripke frames where accessibility is symmetric.

[12] A sequence $x_0, x_1, x_2, \ldots$ of points in a topological space X converges to a point y if, for all open sets U with $y \in U$, there is N such that for all $n \geq N$, we have $x_n \in U$.

the observer of the system. So it is an additional state, 0, which is different from all the system states. But how should we extend the ternary relation, compatibility, and interpretation to this observation state 0? The ternary relation should represent the relation 'the observer looks at system state x', so it models the observer considering a specific system state—or 'situating' themselves in the state space. As stated, this is a binary relation that holds between 0 and any system state x, but formally we can cast it as a ternary relation by requiring $R0xy$ to hold precisely if $x = y$. Conveniently, this is exactly in line with the normality condition. Compatibility and interpretation should be given by the corresponding compatibility and interpretation of the initial state of the system. This is because, as mentioned, the initial state is the current state of the system from which it is analyzed, much like the actual world in possible world reasoning. So the observer takes the compatibility and interpretation of the initial state as their 'beliefs' about the actual state of the system.[13]

An upshot is that, when evaluating the conditional $\varphi \rightsquigarrow \psi$ at the observer's state 0 (rather than a system state), it is true iff any φ-state is a ψ-state—so we get the global conditional mentioned in the intuitive semantics (modulo including the observer's state to the system states). Thus, we make sense of the distinction in the Routley-Meyer semantics between the base world and the non-base worlds—and the corresponding distinction between a global and a local reading of the relevant conditional. Here, it is understood as the distinction between the observer's state and the system states. Correspondingly, the observer takes a global perspective at the whole system and is the one requiring truth-preservation in the definition of consequence, while at the system states the conditional gets the local 'perturbation plus evolution' reading.

Figure 4 summarizes these ideas, which we now formalize in the following definition (further explanations afterward).

[13] An alternative would be to not take 0 to be a new state but the initial state of the system. Then, to satisfy the normality condition, we adjust $Rxyz$: if $x \neq x_0$, it gets the above interpretation, but if $x = x_0$, it is defined as $y = z$. This added clause represents the special status of the initial state as the state from which the system is analyzed. We can also make sense of it in terms of perturbation: We require that (a) there are enough perturbations so that every state can be reached by perturbation from the initial state, and (b) if we reach a state by perturbation from the initial state, then we do not consider any further evolution of the dynamics to evaluate the perturbation conditional. It turns out that our desired soundness and completeness result also works with this alternative. An advantage of this alternative is that it is more parsimonious since it doesn't introduce a new state, but—as helpfully pointed out by a referee—a disadvantage is that not all system states have the 'perturbation plus evolution' interpretation.

relevant model	*dynamical system*
worlds W	state space plus the state of the observer
base world 0	the state of the observer
ternary relation R	limit behavior after some perturbation (for system states) / accessing a state (for the observer state)
normality condition	observer considering a system state
compatibility C	state-compatibility (for system states) / compatibility with initial state (for observer state)
interpretation i	properties of the system (for system states) / properties of the initial state (for the observer state)

Figure 4: Interpreting a dynamical system as a relevant model.

Definition 3. Let $S = (X, x_0, T, A, P, C, i)$ be an interactive topological system. Define the relevant model $\mathsf{M}(S) = (W_S, 0_S, R_S, C_S, i_S)$ induced by the interactive topological system S by:

- $W_S := X \cup \{X\}$
- $0_S := X$
- For $x, y, z \in W_S$, define $R_S xyz$ by:
  - Either $x = 0_S$ and $y = z$,
  - or $x, y, z \in X$ and $\exists a \in A : y \in P(x, a)$ and $\lim_n T^n(y) = z$.
- As a helpful function, define $\widehat{\cdot} : W_S \to X$ by

$$\widehat{x} := \begin{cases} x & \text{if } x \in X \\ x_0 & \text{if } x = 0_S. \end{cases}$$

- For $x, y \in W_S$, define $x C_S y$ by: $\widehat{x} C \widehat{y}$
- For $x \in W_S$ and $p \in \mathsf{At}$, define $i_S(x, p) := i(\widehat{x}, p)$.
- If X carries a partial order $\leq$, define the partial order $x \leq_S y$ on W_S by: either $y \in X$ and $\widehat{x} \leq y$ or $y = 0_S$ and $x = 0_S$.

We formally identify the observer state 0_S with the whole state space X itself. This represents the observer looking at the whole system, and it has the desired consequence that 0_S is a new state, i.e., not in X (by the foundation/regularity axiom in set theory). The ternary relation R_S combines the 'perturbation plus evolution' for system states with the 'looking at a state' for the observer. Finally, the $\widehat{\cdot}$ function situates the observer in the state space: any system state $x \in X$ is already situated, and the observer 0_S situates themselves at the initial state x_0. With this, we express the idea that compatibility and interpretation (as well as order, if available) for the observer state is given by the corresponding notion for the initial state of the system.

Clause	*S as relevant model*	*Intuitive semantics*
$x \Vdash p$	$i_S(x, p) = 1$	property p holds at state x
$x \Vdash \neg\varphi$	$\forall y(xC_S y \Rightarrow y \nVdash \varphi)$	property φ fails in all states compatible with x
$x \Vdash \varphi \wedge \psi$	$x \Vdash \varphi$ and $x \Vdash \psi$	both property φ holds at x and property ψ holds at x
$x \Vdash \varphi \vee \psi$	$x \Vdash \varphi$ or $x \Vdash \psi$	either property φ holds at x or property ψ holds at x
$x \Vdash \varphi \rightsquigarrow \psi$	$\forall y, z(R_S xyz \ \&\ y \Vdash \varphi \Rightarrow z \Vdash \psi)$	whenever we perturb the system from x into a state y with property φ, it will evolve into a state with property ψ
$0_S \Vdash \varphi \rightsquigarrow \psi$	$\forall y(y \Vdash \varphi \Rightarrow y \Vdash \psi)$	every φ-state is a ψ-state

Figure 5: Explicating the informal semantics by regarding the interactive topological system S as a relevant model.

Figure 5 shows that the semantics we get when regarding the system S as the relevant model $\mathsf{M}(S)$ indeed formalizes the intuitive semantics for a logic of dynamical systems from the beginning of this section.[14] Let's illustrate this with our running example.

Example d (Example c continued)**.** Let S be the interactive topological system from example c modeling our population of insects. Let's see what the logic of this dynamical system looks like by regarding it as the

[14] One can interpret relevant models also as time-discrete, possibly non-deterministic dynamical systems, i.e., *labeled transition systems* (LTS). We confine this in a footnote to not digress, but LTSs are an important general model of computing systems used for model checking and concurrent computation (e.g. Baier and Katoen 2008; Winskel and Nielsen 1995). A textbook definition of an LTS (Baier and Katoen 2008, p. 20) is as a tuple $(S, Act, \to, In, AP, L)$ where S is a set of states, Act is a set of actions, $\to \subseteq S \times Act \times S$ is a transition relation (written $x \xrightarrow{a} y$), $In \subseteq S$ is a set of initial states, AP is a set of atomic propositions, and $L : S \times AP \to \{0, 1\}$ is a function. An action a of an LTS is called *idle* if (1) for any state x, we have $x \xrightarrow{a} x$, and (2) if $x \xrightarrow{a} y$, then $x = y$. Now this is reminiscent of the normality condition (1)! In fact, we have the following observation: The positive relevant models $(W, 0, R, i)$ can be viewed as those LTSs $(S, Act, \to, In, AP, L)$ where

- $S = Act$. This is then regarded as the set of worlds W and $\to$ is the ternary relation R. We write $Rxyz$ for $y \xrightarrow{x} z$.
- In is a singleton consisting of an idle action. This action is the base world 0, and being idle precisely means $x \xrightarrow{0} y \Leftrightarrow x = y$.
- $AP = \mathsf{At}$. So AP is the set of propositional atoms and L is the interpretation function i.

To also treat negation, one might extend this by a compatibility relation; cf. asynchronous transition systems (Winskel and Nielsen 1995). This provides an applied interpretation of relevant models as particular LTSs, which are literally applied mathematical structures.

relevant model $\mathsf{M}(S)$. Recall that we consider 'The population is highly populated' (p) and 'The population is healthy' (q).

First, let's consider again the sentence $p \rightsquigarrow q$: saying that if we highly populate the system, it will recover. We claim that it is true at any system state x. Indeed, given states y and z with $R_S xyz$ and $y \Vdash p$, we need to show $z \Vdash q$. Since $R_S xyz$ and x is a system state (i.e., not the observer state), the system in particular evolves from y to z. Since $y \Vdash p$, we have $0.7 < y < 0.9$, so—from what we know about the dynamics—the system will evolve from y to the attracting fixpoint 0.5, so $z = 0.5$, and hence $z \Vdash q$, as needed.

Second, let's consider the sentence $p \rightsquigarrow p$: saying that if we highly populate the system, it will evolve into a highly populated state. Given the preceding reasoning, we expect this to likely be false. Indeed, consider, e.g., the initial state $x_0 = 0.6$. Let's perturb the population by $a = 0.2$. So, say, $y = 0.81 \in P(x_0, a)$. From there, the system converges to $z = 0.5$. Hence $R_S x_0 yz$ and $y \Vdash p$, but $z \nVdash p$, so, indeed, $x_0 \nVdash p \rightsquigarrow p$.

Third, now consider the sentence $q \rightsquigarrow (p \rightsquigarrow p)$. This is an instance of paradox 3, and we would like to see that our system indeed avoids it. So we show that $0_S \nVdash q \rightsquigarrow (p \rightsquigarrow p)$. Indeed, with $x_0 = 0.6$ as before, we have $R_S 0_S x_0 x_0$ and $x_0 \Vdash q$, but $x_0 \nVdash p \rightsquigarrow p$.

Finally, we note that our interpretation also extends to *ordered* relevant models: Let's define an interactive topological system S with a partial order $\leq$ on its state space X to be *ordered* (resp., *star-ordered*) if the induced relevant model $\mathsf{M}(S)$ with the order $\leq_S$ is ordered (resp., star-ordered). An example is the following version of our population of insects.

Example e. We change the interactive topological system S from our population of insects example c: We redefine the perturbation function $P : X \times A \rightrightarrows X$ as $P(x, a) = \{y \in [0, 1] : x + a \leq y \leq 0.55\}$. So when we perturb, we can only support the insect population in growing more quickly (e.g., by providing ideal circumstances), but not much beyond the point it can reach on its own—we can, so to speak, only work with Nature but not against it. We define compatibility as reachability through perturbation, i.e., xCy iff there is a with $y \in P(x, a)$, or, equivalently, $x \leq y \leq 0.55$. And we consider p representing 'The population is above equilibrium' ($i(x, p) = 1$ iff $x > 0.5$), and q representing 'The population is not under-populated' ($i(x, q) = 1$ iff $x > 0.25$). Then S is an ordered interactive topological system.[15] And we still have, e.g., $0_S \nVdash q \rightsquigarrow (p \rightsquigarrow p)$, because

[15]For this, use: If $S = (X, x_0, T, A, P, C, i)$ is an interactive topological system and $\leq$ is a partial order on X, then a sufficient condition for S to be ordered is (a) atomic heredity (if $i(x, p) = 1$ and $x \leq y$, then $i(y, p) = 1$), (b) antitonicity (if yCz and $x \leq y$, then xCz), (c) initial non-perturbability (if $x_0 \leq x$, then, for all a, we have $P(x, a) = \emptyset$), (d) order-reversing perturbation (if $x \leq y$, then $P(y, a) \subseteq P(x, a)$).

$x := 0.5 \Vdash q$ but $x \nVdash p \rightsquigarrow p$ since, by permuting with, e.g., $a = 0.01$, we can choose some $y \in P(x, a) = [0.51, 0.55]$, from which the system converges to $z = 0.5$, so $R_S x_0 yz$ with $y \Vdash p$ but $z \nVdash p$.

9 Soundness and completeness

Let us take stock: we now have a semantics for perturbation conditionals and, more generally, a logic of dynamical systems; and we also have an applied interpretation for some of the abstract models of relevant logic. But, to deliver on the promised contributions, two things are still missing:

1. We want to fully understand the logic of perturbation conditionals and dynamical systems: we want to characterize this logic by its *sound* argument schemes and, ideally, by identifying it as a well-known logic.
2. We want to know that our applied interpretation is *complete*: so far we know that some relevant models come from dynamical systems, but we would like to know that, up to logical equivalence, *all* relevant models are of this form.

We now bring both points home with one theorem, proved in the appendix. Let's first state the theorem and then explain how it delivers:

Theorem 4. *For any relevant model M, there is a simplified interactive topological system S whose induced relevant model is equivalent to M, i.e., their base worlds make true exactly the same sentences. This remains true when adding 'positive', 'ordered', and 'star-ordered'. And if M is finite, S can be chosen to be standard.*[16]

This precisely delivers point 2: not only do dynamical systems induce relevant models, but *all* relevant models come, up to equivalence, from dynamical systems. As for point 1: since all relevant models are governed by the argument schemes of the positive basic relevant logic B^+ (see figure 2), they also are valid for dynamical systems. For example, axiom (A5) yields the following valid argument scheme for systems:

[16]Comments on versions of this result: (1) As mentioned, we could choose the base world as the initial state. (2) One could investigate different notions of (stable) limit behavior, other than plain convergence in defining R_S, e.g., z is a limit point of the orbit of y (in which case the result still works). (3) We could also drop the assumption on systems that the perturbation function is appropriately continuous (i.e., is only a multifunction). (4) The theorem implies that relevant logic cannot 'see' the difference between non-deterministic systems (interpreting M as an LTS as in footnote 14) and deterministic systems (the system S). This is to be expected since, intuitively, this difference need not be observable: a state could have two observationally identical successor states.

> Given a state x, assume (a) whenever we perturb x into a φ-state, it will evolve into a ψ-state, and (b) whenever we perturb x into a φ-state, it will evolve into a χ-state. Then whenever we perturb x into a φ-state, it will evolve into a $\varphi \wedge \chi$-state.

In logical terminology this means that the logic B^+ is sound for dynamical systems. However, to fully understand the logic of dynamical systems, we also want a *complete* description of the sound argument patterns: that anything that is true about all dynamical systems also can be derived from the few sound axioms and rules—which generally is much harder to achieve! It says that these few axioms and rules not only describe the general laws of dynamical systems (soundness), but all general laws of dynamical systems are already described by them (completeness). This is what theorem 4 delivers: different relevant logics can be characterized by different classes of abstract relevant models which, by theorem 4, hence further correspond to different classes of dynamical systems. Concretely, for the example of B^+, we have the following.

Corollary 5. *The logic of positive interactive topological systems is the positive basic relevant logic* B^+*: For a set of formulas* Σ *and a formula* φ*, we have that* φ *is derivable from* Σ *in the logic* B^+ *iff, in any positive interactive topological system* S*, the formula* φ *is true at the observer state* 0_S *whenever all formulas of* Σ *are true at* 0_S*.*

10 Stronger relevant logics

So we have delivered the two main contributions of this paper and thus established a bridge between dynamical systems and relevant logics. We did so for a broad class of dynamical systems and a broad class of relevant models. This was to include all potential systems and provide an applied semantics for all relevant models. The obvious next item in an agenda for future work is: how do interesting *sub*classes of systems correspond to *stronger* relevant logics?

Relevant logicians have come up with systems stronger than B, some of which have been applied for various purposes: e.g., to model common knowledge (Punčochář and Sedlár 2021) or justification (Standefer forthcoming), as underlying logics for non-classical formal theories of arithmetic (R. K. Meyer 1976), formal semantics for languages expressing their own transparent truth predicates (Beall 2009), set theories with unrestricted comprehension principles (R. Routley 1979; Weber 2021).

What we need is a *correspondence theory*. This is well-known from modal logic (Blackburn, Rijke, and Venema 2001): one starts with the basic normal modal logic K, and shows that it is sound and complete with respect to Kripke models (W, R, i) which don't have any constraints on the binary accessibility relation R. Then one identifies constraints on R: e.g., reflexivity or transitivity. And one shows that these constraints *correspond* to formulas of the logic in the sense that they are valid on the so constrained class of models: e.g., $\Box\varphi \to \varphi$ or $\Box\varphi \to \Box\Box\varphi$, respectively. This helps to find and illuminate the intended applied interpretation of the logical operators by relating structural to logical intuitions. Now a correspondence theory for relevant logic also has been developed (Priest and Sylvan 1992; Restall 1993, 2000, sec. 11.5). In the remainder of this section, we start to thus investigate the logic of subclasses of dynamical systems.

In relevant correspondence theory, one gets relevant logics stronger than B by imposing constraints on the ternary R, which then validate more principles than those of B. However, the constraints are often more cumbersome than in modal logic, and they may involve the Routley star $*$ and/or the hereditary ordering $\leq$. Examples from the literature (taken from Priest (2008, ch. 10), notation adjusted) include:

1. If $Rxyz$, then Rxz^*y^*.
2. If there is a $w \in W$ such that $Ruvw$ and $Rwxy$, then there is a $z \in W$ such that $Ruxz$ and $Rvzy$.
3. If there is a $w \in W$ such that $Ruvw$ and $Rwxy$, then there is a $z \in W$ such that $Rvxz$ and $Ruzy$.
4. If $Rxyz$, then there is a $w \in W$ such that $Rxyw$ and $Rwyz$.
5. If $Rxyz$, then there is a $w \in W$ such that $x \leq w$ and $Rywz$.

These validate, respectively:

a. $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$ (Contraposition)
b. $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ (Suffixing)
c. $(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$ (Prefixing)
d. $(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$ (Contraction)
e. $(\varphi \to ((\varphi \to \psi) \to \psi)$ (Assertion)

The logic with a–c on board is called TW. With d–e added, it's known as R: possibly the most established relevant logic. It can be proved to have the Variable Sharing Property, so all weaker systems have it, too (Priest 2008, pp. 205–6).

Do these conditions make sense in the dynamical systems interpretation? Let $S = (X, x_0, T, A, P, C, i)$ be an interactive topological system. We may assume it's ordered by $\leq$, to talk about condition 5. To fix some useful notation, write $x \dashrightarrow y$ for $\exists a \in A : y \in P(x, a)$, i.e., the system in state x can be perturbed to state y. Write $x \rightarrow y$ for $\lim_n T^n(x) = y$, i.e., the system in state x will evolve to state y. Some reasonable assumptions on perturbability are the following (explanations afterward):

i. If $x \dashrightarrow y$ and $y \rightarrow z$, then $x \dashrightarrow z$. $(\dashrightarrow\rightarrow\Rightarrow\dashrightarrow)$
ii. If $x \rightarrow y$ and $y \dashrightarrow z$, then $x \dashrightarrow z$. $(\rightarrow\dashrightarrow\Rightarrow\dashrightarrow)$
iii. If $x \dashrightarrow y$ and $y \dashrightarrow z$, then $x \dashrightarrow z$. $(\dashrightarrow\dashrightarrow\Rightarrow\dashrightarrow)$
iv. If $x \rightarrow y$, then $y \dashrightarrow x$. $(\rightarrow\Rightarrow\dashleftarrow)$
v. If $x \dashrightarrow y$, then $x \leq y$. $(\dashrightarrow\Rightarrow\leq)$
vi. $x \dashrightarrow x$. $(\Longrightarrow\dashrightarrow)$

Here, i and ii say that first perturbing and then converging, or vice versa, can also be realized by one perturbation. iii says that perturbability is transitive (e.g., by chaining the perturbations).[17] iv says that convergence can be reversed by perturbation. v says that we can only perturb along the order of the system. And vi modestly demands that perturbability be reflexive: that there is the trivial perturbation of not doing anything.

As we'll now see, these assumptions go a *long* way towards validating the model conditions 1–5 for the relevant logic R. For reasons of space, we won't discuss condition 1 as it only has to do with negation and hence compatibility, whereas our prime concern is with the conditional and hence the ternary relation. And we omit looking separately at the base world. The aim here is to illustrate ideas, not formal correctness.

For condition 2, consider figure 6.[18] The antecedent of the implication shows that $R_S uvw$ (i.e., $u \dashrightarrow v$ and $v \rightarrow w$) and $R_S wxy$ (i.e., $w \dashrightarrow x$ and $x \rightarrow y$). Using $z := y$, we claim $R_S uxz$ and $R_S vzy$: indeed,

- $u \dashrightarrow x$ because $u \dashrightarrow v \rightarrow w \dashrightarrow x$ implies, by i, $u \dashrightarrow w \dashrightarrow x$, which, by iii, implies $u \dashrightarrow x$
- $x \rightarrow z$ by assumption
- $v \dashrightarrow z$ because $v \rightarrow w \dashrightarrow x \rightarrow y$ implies, by ii, $v \dashrightarrow x \rightarrow y$, which, by i, implies $v \dashrightarrow y$, and
- $z \rightarrow y$ because, since $\lim_n T^n(x) = y$, continuity implies $T(y) = y$, so $\lim_n T^n(y) = y$.

We argue similarly for condition 3. For condition 4, if $R_S xyz$, choose $w := z$: then, trivially, $R_S xyw$, and, since $y \rightarrow z$, iv implies, $w \dashrightarrow y$, so $R_S wyz$. For condition 5, if $R_S xyz$, choose $w := y$: since $x \dashrightarrow y$, v implies $x \leq w$, and we have $y \dashrightarrow w$ (by vi) and $w \rightarrow z$ (by assumption), so $R_S ywz$.

[17] This is the first step toward a monoid/group structure on the set of perturbations A, so one can speak—in the mathematical sense—of it acting on the state space X.

[18] Much inspired by the notation of Priest (2008, p. 194).

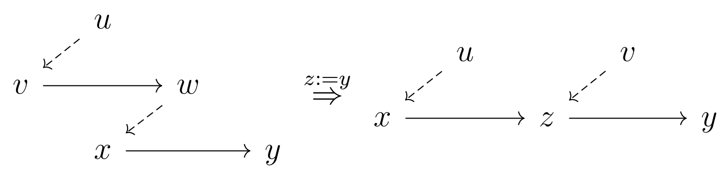

Figure 6: Validating the suffixing condition in dynamical systems.

11 Conclusion

We have shown that dynamical systems provide a natural interpretation for the semantics of relevant logic. Then relevant logic, conversely, provides the tools for reasoning about dynamical systems, in particular by capturing, via the logical validities involving the relevant conditional, the laws governing the systems.

Further work should continue section 10: investigating the applied semantics for restricted classes of dynamical systems and correspondingly stronger relevant logics. Similarly, one may ask which relevant logics can be realized over a particular class of systems (e.g., edge shifts, ergodic systems, etc.) or maybe even in a single system (say a continuous dynamics on the real line). Another question is whether the two constructions from systems to models and back can be seen as category-theoretic functors—maybe even adjoint ones. A further exciting avenue is to explore if other relevant connectives like fusion or the Ackermann truth constant (Standefer 2022) and the related linear logic (Allwein and Dunn 1993) can be given a dynamical system interpretation—thus also providing a richer language for systems. Moreover, one can investigate a counterfactual version of our relevant conditional (Mares and Fuhrmann 1995) by considering the *smallest* or *minimal*—rather than any—perturbation rendering the antecedent true.[19]

[19]Many thanks to two anonymous referees who provided detailed comments that improved the paper. For helpful discussions, we are also grateful to Hannes Leitgeb and the audience of the Work in Progress Talk Series at the *Munich Center for Mathematical Philosophy*, LMU Munich. Franz Berto's research is funded by a Leverhulme Trust Research Project Grant RPG-2023-236, *What If...? Knowing by Imagining [WIKI]: the Logic and Rationality of Imagination.* Part of Levin Hornischer's work was done within the project 'Foundations of Analogical Thinking' (Project No. 322-20-017) of the research program 'PhDs in the Humanities', financed by the *Dutch Research Council* (NWO).

Appendix: Proof of theorem 4

Given a relevant model M, we build the interactive topological system $S = \mathsf{S}(M)$, whose induced relevant model is equivalent to M, by using the perspective on relevant models as LTSs (footnote 14) and the toolbox of symbolic dynamics (Lind and Marcus 1995).

We think of $Rxyz$ in M as a transition from state y to state z with label x, written $y \xrightarrow{x} z$. We call y the *starting state* of $Rxyz$ and z the *ending state* of $Rxyz$. If the starting states and the ending states match, we can chain together transitions to obtain a path (or trajectory):

$$\dots \xrightarrow{x_{-2}} z_{-2} = y_{-1} \xrightarrow{x_{-1}} z_{-1} = y_0 \underline{\xrightarrow{x_0}} z_0 = y_1 \xrightarrow{x_1} z_1 = y_2 \xrightarrow{x_2} \dots,$$

where the underlined arrow denotes the time step 0. Formally, a *two-sided infinite path* (or just *path*) in M is a function $t : \mathbb{Z} \to R$ such that, for all $n \in \mathbb{Z}$, the ending state of $t(n)$ is identical to the starting state of $t(n+1)$. (Mnemonic: t as in trajectory.)

Some useful terminology: For $x \in W$, write $\overline{x}$ for the path

$$\dots \xrightarrow{0} x \xrightarrow{0} x \underline{\xrightarrow{0}} x \xrightarrow{0} x \xrightarrow{0} \dots .$$

Call a path t *pure* if there is $x \in W$ such that $t = \overline{x}$. For a path t, write t_0 for the starting state of $t(0)$ (i.e., if $t(0) = y \xrightarrow{x} z$, then $t_0 = y$).

Now, the idea to build system S is that the states of S are the paths in M and the dynamics is the shift operator (moving all transitions once to the right). Formally:

Definition 6. Let $M = (W, 0, R, C, i)$ be a relevant model. Define a system:

- X is the set of all two-sided infinite paths in M. The topology on X is the subspace topology of the product topology on $R^{\mathbb{Z}}$ where R carries the discrete topology.
- $x_0 := \overline{0}$.
- $\sigma : X \to X$ is the shift operator. It maps a path t to the the path $\sigma(t)$ defined by $\sigma(t)(n) = t(n+1)$.
- $A := X$
- $v \in P(t, u)$ iff there are $x, y, z \in W$ such that $t_0 = x$, $u = \overline{y}$, $Rxyz$ (i.e., $y \xrightarrow{x} z$) and v is the path

$$\dots \xrightarrow{0} y \xrightarrow{0} y \underline{\xrightarrow{x}} z \xrightarrow{0} z \xrightarrow{0} \dots .$$

- Compatibility: tC_0u iff $t_0 C u_0$.
- Interpretation: $i_0(t, p) := i(t_0, p)$.

Call $\mathsf{S}(M) = (X, x_0, \sigma, A, P, C_0, i_0)$ the interactive topological system *induced by* the relevant model M. (Lemma 7 below shows that this is well-defined.)

If M was an ordered relevant model with order $\leq$, define the order $\sqsubseteq$ (well-defined by lemma 10 below) on $\mathsf{S}(M)$ by

- $t \sqsubseteq u$ iff (a) $t = u$, or (b) $t_0 < u_0$, or (c) $t_0 = u_0$ and $u = \overline{u_0}$.

So compatibility and interpretation in the system are compatibility and interpretation, respectively, at time 0 in the model. And perturbing a path t by a (pure) path u yields the 'merged' path v above.

In the remainder, we check that $S = \mathsf{S}(M)$, when regarded as a relevant model, is indeed equivalent to M. Let's start by checking that S is an interactive topological system.

Lemma 7. *In the setting of definition 3, $\mathsf{S}(M)$ is a simplified interactive topological system. It is standard if M is finite.*

Proof. Write $M = (W, 0, R, C, i)$ and $\mathsf{S}(M) = (X, x_0, \sigma, A, P, C_0, i_0)$. The opens of X are given by arbitrary unions of finite intersections of sets of the form U^n_r, which contain all paths whose n-th component is $r \in R$. So X is Hausdorff (if $t \neq u$ differ at position n, they are separated by the disjoint opens $U^n_{t(n)}$ and $U^n_{u(n)}$) and σ is continuous (the σ-preimage of U^n_r is U^{n+1}_r). By construction, $\mathsf{S}(M)$ is simplified. Next, σ clearly is bijective; and if M is finite, also R is finite, hence compact, so, by Tychonoff's theorem, the product space $R^{\mathbb{Z}}$ is compact, so, since X is closed, it also is compact.

It remains to show that $P : X \times X \rightrightarrows X$ is appropriately continuous. For $x, y \in W$, write

$$\begin{aligned} R^x &:= \{r \in R : \text{the label of } r \text{ is } x\} \\ R_y &:= \{r \in R : \text{the starting state of } r \text{ is } y\}. \end{aligned}$$

Closed-valued: Given $t, u \in X$, if u is not pure, $P(t, u) = \emptyset$ is closed. If $u = \overline{y}$ is pure, write $x := t_0$, then

$$\begin{aligned} P(t,u) &= \left\{ v \in X : \exists z \in W : Rxyz \text{ and } v = \dots y \xrightarrow{0} y \xrightarrow{x} z \xrightarrow{0} z \dots \right\} \\ &= \left\{ v \in X : \forall 0 \neq n \in \mathbb{Z} : v(n) \in R^0, v(0) \in R^x, v(0) \in R_y \right\} \end{aligned}$$

which is closed qua intersection of closed sets.

Upper hemicontinuous: For $t, u \in X$, let $V \subseteq X$ be open with $P(t, u) \subseteq V$. If u is not pure, consider

$$U_0 := \{(t', u') \in X \times X : \exists n \in \mathbb{Z} : u'(n) \in R \setminus R^0\}.$$

This is open with $(t, u) \in U_0$ and, for all $(t', u') \in U_0$, we have $P(t', u') = \emptyset \subseteq V$, as needed. So let $u = \overline{y}$ be pure. Write $x := t_0$. Define

$$U_y^x := \Big\{(t', u') \in X \times X : t'(0) \in R_x \text{ and } u'(0) \in R_y\Big\}.$$

Then U_y^x is open with $(t, u) \in U_y^x$ and, for $(t', u') \in U_y^x$, it's easily shown that $P(t', u') \subseteq P(t, u)$, and hence $P(t', u') \subseteq V$, as needed.

Closed-domain: To show that $\{(t, u) : P(t, u) \neq \emptyset\}$ is closed, let $t, u \in X$ with $P(t, u) = \emptyset$ and find an open $U \subseteq X \times X$ such that $(t, u) \in U$ and for all $(t', u') \in U$, we have $P(t', u') = \emptyset$. If u is not pure, we can choose U_0 as above. If $u = \overline{y}$, write $x := t_0$, and we can choose U_y^x as above (if $(t', u') \in U_y^x$, then $P(t', u') \subseteq P(t, u) = \emptyset$). □

The next lemma characterizes convergence of the dynamics in $\mathsf{S}(M)$.

Lemma 8. *In the setting of definition 3, the following are equivalent for $t, u \in X$:*

1. $\lim_n \sigma^n(t) = u$
2. *u is a constant path which also is the tail of t: i.e., for all $n \in \mathbb{Z}$, $u(n) = u(0)$ and there is N such that, for all $n \geq N$, $t(n) = u(0)$.*

Proof sketch. (1)⇒(2). Assume $\lim_n \sigma^n(t) = u$. To show that u is constant, let $k \in \mathbb{Z}$ and show $u(k) = u(k+1)$. Consider the open set $U = U^k_{u(k)} \cap U^{k+1}_{u(k+1)}$. Since $u \in U$ and $\sigma^n(t)$ converges to u, there is N such that, for all $n \geq N$, $\sigma^n(t) \in U$. Since $\sigma^N(t) \in U$ and $\sigma^{N+1}(t) \in U$,

$$\sigma^N(t)(k+1) = u(k+1) \qquad\qquad \sigma^{N+1}(t)(k) = u(k).$$

Now, since $\sigma^{N+1}(t)(k) = \sigma^N(t)(k+1)$, we get $u(k) = u(k+1)$. To show that t has tail $u(0)$, consider $U^0_{u(0)}$.

(2)⇒(1). Assume u is a constant path which is also the tail of t. If U^k_r is an open neighborhood of u (generalizing to any open neighborhood is not too difficult), let N be the start of the u-tail of t, then, for $n \geq N$, we have $\sigma^n(t)(k) = t(n+k) = u(k) = r$, so $\sigma^n(t) \in U^k_r$. □

Now the key lemma is to relate truth in the original relevant model M to truth in the relevant model induced by the system $\mathsf{S}(M)$.

Lemma 9. *Let M be a relevant model. Then, for all states t of $\mathsf{M}(\mathsf{S}(M))$ and for all formulas φ,*

$$\mathsf{M}(\mathsf{S}(M)), t \Vdash \varphi \Leftrightarrow M, \widehat{t}_0 \Vdash \varphi.$$

Proof. Write $M = (W, 0, R, C, i)$ and $S := \mathsf{S}(M) = (X, x_0, \sigma, A, P, C_0, i_0)$. The proof is by induction on φ.

For atomic φ, by definition, $t \Vdash \varphi$ iff $i_0(\widehat{t}, \varphi) = 1$ iff $i(\widehat{t}_0, \varphi) = 1$ iff $\widehat{t}_0 \Vdash \varphi$.

For $\neg\varphi$, we have to show

$$\Big(\forall u \in W_S : t(C_0)_S u \Rightarrow u \nVdash \varphi\Big) \text{ iff } \Big(\forall y \in W : \widehat{t}_0 C y \Rightarrow y \nVdash \varphi\Big).$$

($\Rightarrow$) Let $y \in W$ with $\widehat{t}_0 C y$. Consider $\overline{y} \in X$. Then $\widehat{t} C_0 \overline{y}$, so $t(C_0)_S \overline{y}$, and hence, by assumption, $\overline{y} \nVdash \varphi$. By induction hypothesis, $y = \overline{y}_0 \nVdash \varphi$, as needed.

($\Leftarrow$) Let $u \in W_S$ with $t(C_0)_S u$. So $\widehat{t} C_0 \widehat{u}$, so $\widehat{t}_0 C \widehat{u}_0$. By assumption, $\widehat{u}_0 \nVdash \varphi$. By induction hypothesis, $u \nVdash \varphi$, as needed.

For $\varphi \wedge \psi$ and $\varphi \vee \psi$, this is immediate by induction hypothesis.

For $\varphi \rightsquigarrow \psi$, we have to show

$$\Big(\forall u, v \in W_S : R_S tuv \text{ and } u \Vdash \varphi \Rightarrow v \Vdash \psi\Big)$$
$$\text{iff } \Big(\forall y, z \in W : R\widehat{t}_0 yz \text{ and } y \Vdash \varphi \Rightarrow z \Vdash \psi\Big).$$

($\Rightarrow$) Let $y, z \in W$ with $R\widehat{t}_0 yz$ and $y \Vdash \varphi$. We need to show $z \Vdash \psi$.

If $t = 0_S$, then $\widehat{t}_0 = 0$, so $R\widehat{t}_0 yz$ implies $y = z$, so $\overline{y} = \overline{z}$ and $R_S t\overline{y}\overline{z}$. By induction hypothesis, $\overline{y} \Vdash \varphi$, so the assumption implies $\overline{z} \Vdash \psi$, hence, again by induction hypothesis, $z \Vdash \psi$.

So let $t \neq 0_S$, so $\widehat{t} = t \in X$. Define $x := t_0$ and v as the path starting with y moving via x to z at time 0 and then staying there. Then, by definition, $v \in P(t, \overline{y})$, and, by lemma 8, $\lim_n \sigma^n(v) = \overline{z}$. Hence, by definition of R_S, we have $R_S tv\overline{z}$. Since $\widehat{v}_0 = y \Vdash \varphi$, the induction hypothesis implies $v \Vdash \varphi$. So the assumption implies $\overline{z} \Vdash \psi$. Hence, by induction hypothesis, $z = \widehat{\overline{z}}_0 \Vdash \psi$, as needed.

($\Leftarrow$) Let $u, v \in W_S$ with $R_S tuv$ and $u \Vdash \varphi$. Show $v \Vdash \psi$.

If $t = 0_S$, then $R_S tuv$ implies $u = v$. Hence $\widehat{t}_0 = 0$ and $\widehat{u} = \widehat{v}$ and hence also $\widehat{u}_0 = \widehat{v}_0$. So $R\widehat{t}_0\widehat{u}_0\widehat{v}_0$. By induction hypothesis, $\widehat{u}_0 \Vdash \varphi$, so the assumption implies $\widehat{v}_0 \Vdash \psi$, so, again by induction hypothesis, $v \Vdash \psi$, as needed.

So assume $t \neq 0_S$. Then $R_S tuv$ implies $t, u, v \in X$ and there is $a \in A = X$ with $u \in P(t, a)$ and $\lim_n \sigma^n u = v$. Since $u \in P(t, a)$, there are $x, y, z \in W$ such that $t_0 = x$, $a = \overline{y}$, $Rxyz$, and u is the path starting with y moving via x to z at time 0 and then staying there. Since $\lim_n \sigma^n u = v$, lemma 8 implies $v = \overline{z}$. Moreover, since $u \Vdash \varphi$, the induction hypothesis yields $y = \widehat{u}_0 \Vdash \varphi$. Since $R\widehat{t}_0 yz$ (since $t_0 = x$), the assumption implies $z \Vdash \psi$, so, again by induction hypothesis, we have, since $\widehat{v}_0 = \overline{z}_0 = z$, that $v \Vdash \psi$, as needed. □

Finally, we need that the construction of the system S from a relevant model M preserves (star-) orderedness.

Lemma 10. *If M is a (star-) ordered relevant model, then $\mathsf{S}(M)$ is a (star-) ordered interactive topological system.*

Proof. Write $M = (W, 0, R, C, \leq, i)$ and $S := \mathsf{S}(M) = (X, x_0, \sigma, A, P, C_0, \sqsubseteq, i_0)$. It is straightforward to verify that $\sqsubseteq$ is a partial order on X. For heredity, assume t and u are in $\mathsf{M}(S)$ with $t \Vdash \varphi$ and $t \sqsubseteq_S u$. By definition of $\sqsubseteq$, if $t \sqsubseteq_S u$, then $\widehat{t} \sqsubseteq \widehat{u}$, which in turn implies $\widehat{t}_0 \leq \widehat{u}_0$. By lemma 9, $\widehat{t}_0 \Vdash \varphi$. So, since M is ordered, $\widehat{u}_0 \Vdash \varphi$. So by lemma 9, $u \Vdash \varphi$, as needed.

Finally, assuming M to be star-ordered, we show that also S is star-ordered, i.e., $(C_0)_S$ is symmetric, serial, and convergent. Symmetric: If $t(C_0)_S u$, then $\widehat{t} C_0 \widehat{u}$, so $\widehat{t}_0 C \widehat{u}_0$, so $\widehat{u}_0 C \widehat{t}_0$, so $\widehat{u} C_0 \widehat{t}$, so $u(C_0)_S t$. Serial: Given t, let y be such that $\widehat{t}_0 C y$. Then $\widehat{t} C_0 \overline{y}$, so $t(C_0)_S \overline{y}$. Convergent: Given t, assume $t(C_0)_S u$ for some u. So $x := \widehat{t}_0$ is C-compatible with something (namely $\widehat{u}_0$), so let x^* be the $\leq$-greatest element C-compatible with x. We claim that $u := \overline{x^*}$ is the $\sqsubseteq_S$-greatest element $(C_0)_S$-compatible with t. Indeed, we have $t(C_0)_S u$ since $\widehat{t}_0 = x C x^* = \widehat{u}_0$. And if v is such that $t(C_0)_S v$, we show that $v \sqsubseteq_S u$. Indeed, since $t(C_0)_S v$, we have $x = \widehat{t}_0 C \widehat{v}_0 =: w$, so $w \leq x^*$. If $w < x^*$, we have $\widehat{v} \sqsubseteq u$, so, since $u \in X$, also $v \sqsubseteq_S u$. And if $w = x^*$, we have, since $\widehat{v}_0 = w = x^* = u_0$ and $u = \overline{u_0}$, that $\widehat{v} \sqsubseteq u$, so again $v \sqsubseteq_S u$. □

Now our desired theorem 4 follows:

Proof of theorem 4. Let M be a relevant model. Choose S to be the simplified interactive topological system $\mathsf{S}(M)$ from definition 6. If M is finite, S is standard. Writing 0 for the base world of M, the base world of the relevant model induced by S (i.e., $\mathsf{MS}(M)$) is 0_S. Then lemma 9 implies, for all φ, that $0_S \Vdash \varphi$ iff $0 = \widehat{0_S}_0 \Vdash \varphi$.

If M was a positive relevant model, we could do the same construction of $\mathsf{S}(M)$ and proof of lemma 9 but ignore compatibility and negation. Then S would be a positive simplified interactive topological system whose induced relevant model is equivalent to M. If M is ordered, lemma 10 implies that S is ordered, and they still are equivalent by lemma 9. The same holds when replacing 'ordered' by 'star-ordered'. □

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