Stone Duality

Connecting Algebra and Topology via Logic

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Link to course website:
levinhornischer.github.io/StoneDuality_ESSLLI/

Welcome!

- This course is an introduction to Stone duality.
- This is an exciting area of logic at the intersection of math, computer science, and philosophy.
- Stone's theorem says that certain algebras (Boolean algebras) are in a precise sense equivalent to certain topological spaces (now called Stone spaces).
- Behind this mathematical fact lies a deep conceptual insight: that the algebraic and the spatial perspective are just two sides of the same coin!
- Understanding this is the goal of the course.

Practicalities

- Schedule: in the first week of ESSLLI 2026 (August 3–7, 2026), from 09:00 to 10:30 in room C206.
- About me: I'm Levin Hornischer (he/him), an assistant professor at the Munich Center for Mathematical Philosophy (MCMP) at LMU Munich. Before that, I was at the Institute for Logic, Language and Computation (ILLC) at the University of Amsterdam, where I also did my PhD.
- Comments and feedback are most welcome: just email me at `Levin.Hornischer@lmu.de`.
- Please interrupt for questions any time!

Materials I

- There is a website for the course:
  `levinhornischer.github.io/StoneDuality_ESSLLI/`

- These slides are intended as handouts for the lecture (hence more cluttered than presentation slides—sorry!).
- I have given a full semester course on Stone duality, on which this course is based. You find the lecture notes linked on the course website.
- Prerequisites: An introductory course in logic (specifically classical propositional logic) and some familiarity with mathematics (ideally, but not necessarily, having seen elementary concepts of topology and algebra).

Materials II

- A wonderful recent textbook on Stone duality is [GG24] and further great textbooks on—or closely related to—Stone duality include [BD75; DP02; Vic89; GH08; Giv14; Grä11; Grä03; AJ94].
- Research monographs on duality theory and related topics include [Joh82; Gie+03; DST19; Gou13; PP12; ORR15].
- Extensions of Stone duality from propositional to first-order logic include [Mak87; Lur20; AF13].
- Stone duality also is an excellent pathway into category theory [Lei14; Smi25], and in particular categorical logic [Pit01] and topos theory [Vic07].

Plan

1. Introduction and motivation: the idea of a duality
2. Boolean algebras
3. Stone spaces
4. Stone's theorem and proof idea
5. Applications and generalizations

Outline

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The general idea of a duality

- The general idea of a duality is that there are two, in a sense complementary perspectives on the same underlying phenomenon.
- Metaphorically speaking, there are two sides (the perspectives) of the same coin (the phenomenon).
- This occurs throughout the sciences. We consider four brief examples. (Also see [Geh09; Abr91; Vic89].)

Duality in mathematics

- Now we all know it, but it is remarkable: there is a close correspondence between algebra and geometry.
- Algebraic objects, like the polynomial $x^2 + y^2 = 1$, correspond to spatial objects, in this case to the unit circle.
- We can solve geometric problems, e.g., which ratios and angles can be constructed with a compass and straightedge, by translating them to the algebraic side and then using algebraic methods (here Galois theory).
- The duality of algebra and geometry is at the heart of algebraic geometry and bears quite some similarities to Stone duality [Joh82; MM92; Vic07].

Duality in physics

- On the one hand, a physical system is described by its state space, i.e., the collection of states that the system could be in.
- On the other hand, the system is also described by the observations we can make about it.
- We would hope that these two perspectives on the system are 'dual' to each other:
  - The states should be determined by the observations that they give rise to; and
  - the observations should be determined by the states that give rise to them [e.g. Str08, p. 24].
- The formal duality (e.g., Gelfand–Naimark) is different in detail from Stone duality but similar in spirit. (For more, see, e.g., [Rue11; RBW15; WW23; DB25].)

Duality in computer science

- Computer programs are written in a programming language. Much like for sentences in a natural language, we can ask what their meaning is.
- The meaning of a program is called its denotation and is, roughly, the input-output function that it computes.
- But we can also describe the program by its program logic:
  - e.g., given an even number as input, the program outputs a prime number.
  - More generally: Hoare triples 'if currently P holds, then running the program C will result in Q being satisfied'.
- The far-reaching insight of Abramsky [Abr91] was that we again should have a duality between the denotational semantics and the program logic.

Duality in philosophy

- The central question of philosophy of language is: what is the meaning of sentences?
- One prominent view is possible worlds semantics: The meaning of a sentence is the set of possible worlds in which the sentence is true.
- On a broadly inferentialist position, we describe sentence meanings by how they relate to each other: which implies which, what conjunctions are, etc. This data is recorded in the set of all sentence meanings together with the implication relation, conjunction, negation, etc.—aka the algebra of propositions.
- Duality talk is surprisingly scarce in philosophy, but given the success in other fields it holds much promise!

Summary

Field	*Phenomenon*	*Space*	*Algebra*
Mathematics	Geometry	Spatial object	Polynomial
Physics	Physical system	State space	Observations
Computer Science	Program	Denotation	Program logic
Philosophy	Meaning	Possible worlds	Propositions

- Algebra: Set of observations, propositions, etc. with operations (e.g., conjunction $\wedge$) that satisfy certain laws (e.g., $a \wedge b = b \wedge a$).
  - A specific type of such an algebra we will make precise as a Boolean algebra.
- Space: Set of points (e.g., states, possible worlds, etc.) with some spatial structure, e.g., a notion of distance between points.
  - A specific type of such a space we will make precise as a Stone space, which is a particular kind of topological space.

Key idea of duality

- Duality: A bijective correspondence between the class of spaces and the class of algebras under consideration, i.e.,
  - a method to translate a space into an algebra, and
  - a method to translate an algebra into a space,
  - such that translating forth and back is isomorphic to where one started.
- In this sense of intertranslatability, the spatial perspective is equivalent to the algebraic perspective and both describe the same phenomenon.
- In our specific case, Stone's theorem makes this precise as: the category of Boolean algebras is dually equivalent to the category of Stone spaces.
- Such a duality not only provides a deep conceptual link between the two perspective. It also can help practically by translating a problem from one side to the other, where different methods and intuitions are available (cf. again geometry and algebra).

Example: possible worlds vs propositions I

We do not have the precise concepts yet, but let's sketch how the duality looks like in the philosophy example.

- Translation from spaces to algebras:
  - Assume a space X of possible worlds. According to possible worlds semantics, a proposition a is a set of possible worlds, i.e., $a \subseteq X$.
  - Conversely, is every subset $a \subseteq X$ a proposition? This is not commonly discussed, but we will see: only if a is "closed under similarity" (technically: a clopen subset in the topology on X).
  - Proposition a implies proposition b iff in all worlds $x \in X$ in which a is true, also b is true, i.e., $a \subseteq b$.
  - The conjunction of two propositions a and b simply is the intersection $a \cap b$. Similarly for other operations.
  - So the space X translates to the algebra $\mathsf{Clp}(X) = (A, \subseteq, \cap, \ldots)$, where A is the set of all $a \subseteq X$ that are "closed under similarity".

Example: possible worlds vs propositions II

- Translation from algebras to spaces:
  - Assume an algebra $A = (A, \leq, \wedge, \dots)$ of propositions.
  - According to Ersatzism, given a language/propositions, we can construct 'substitute possible worlds' as maximally consistent sets of propositions.
  - So an Ersatz possible world is a set $F \subseteq A$ such that
    (1) F is closed under implication and conjunction and does not contain the logical contradiction $\perp$, and
    (2) for any other such set $G \subseteq A$, if $F \subseteq G$, then $F = G$.

    (We will call such an F a maximal filter.)
  - Let $\mathsf{Spec}(A)$ be the space of all Ersatz possible worlds (we will see that there is a natural way to define a topology on $\mathsf{Spec}(A)$).

Example: possible worlds vs propositions III

- Translating a space X of possible worlds to the algebra $\mathsf{Clp}(X)$ and then back to Ersatz space $\mathsf{Spec}(\mathsf{Clp}(X))$ yields a space "isomorphic" to X.
  - We need, in particular, a bijection $f : X \to \mathsf{Spec}(\mathsf{Clp}(X))$.
  - We map a world x to the set $f(x) = \{a \in \mathsf{Clp}(X) : x \in a\}$ of all (possible worlds semantics) propositions that x makes true.
- Translating an algebra A to Ersatz worlds $\mathsf{Spec}(A)$ and then back to the algebra $\mathsf{Clp}(\mathsf{Spec}(A))$ yields an algebra isomorphic to A.
  - We need, in particular, a bijection $h : A \to \mathsf{Clp}(\mathsf{Spec}(A))$.
  - We map a proposition a to the set $h(a) = \{F \in \mathsf{Spec}(A) : a \in F\}$ of Ersatz worlds that make true a.

Now we develop these ideas formally.

Outline

Overview of the section

- We formally define Boolean algebras.
- We motivate how they describe sets of propositions or binary experiments.
- We consider three important examples: the two-element Boolean algebra, powerset Boolean algebras, and the Lindenbaum–Tarski algebra.
- Finally, we define 'translations' between Boolean algebras, i.e., Boolean algebra homomorphisms.

Boolean algebra I

Definition

A Boolean algebra is a structure $(A, \vee, \wedge, \neg, \bot, \top)$ where A is a set and $\vee, \wedge : A \times A \to A$ and $\neg : A \to A$ and $\bot, \top \in A$ are such that for $a, b, c \in A$:

1. Lattice axioms:
   1.1 Commutativity: $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$
   1.2 Associativity: $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
   1.3 Idempotence: $a \vee a = a$ and $a \wedge a = a$
   1.4 Absorption: $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.
2. Distributivity:
   2.1 $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
   2.2 $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
3. Bounds: $a \wedge \bot = \bot$ and $a \vee \top = \top$
4. Complements: $a \wedge \neg a = \bot$ and $a \vee \neg a = \top$.

Boolean algebra II

Definition (continued)

We often just write A for the Boolean algebra $(A, \vee, \wedge, \neg, \bot, \top)$. For $a, b \in A$, we define

$$a \leq b :\Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b.$$

This is a partial order on A (i.e., $\leq$ is a reflexive, transitive, and antisymmetric relation on A).

(Exercise: Prove that $\leq$ indeed is a partial order. Prove the equivalence $a \wedge b = a \Leftrightarrow a \vee b = b$.)

Comments I

- Intuition:
  - One can think of A as a set of propositions, observations, binary experiments, etc.
  - for which there are 'logical' operations $\vee, \wedge, \neg, \bot, \top$.
  - E.g., given propositions (or experiments) a and b, there also is the proposition (or experiment) $a \wedge b$, which is true (or comes out positive) iff both a and b are true (or come out positive).
  - The order $\leq$ describes implications between propositions (or experiments). After all, if a is equivalent to $a \wedge b$ (i.e., $a = a \wedge b$), this just says that a implies b.
  - The propositions (or experiments) should behave 'classically' in the sense that, e.g., they commute $a \vee b = b \vee a$, are exhaustive $a \vee \neg a = \top$, etc.

Comments II

- Names of the axioms:
  - The bound axioms say that $\bot$ is the $\leq$-least element and that $\top$ is the $\leq$-greatest element.
  - Distributivity: If we write $\wedge$ as $\times$ (multiplication) and $\vee$ as $+$ (addition) axiom 2.2 is well-known distributivity $x(y+z) = xy + xz$.
- Some axioms are superfluous:
  - e.g., distributivity is self-dual, i.e., (2.1) is equivalent to (2.2).
- More general structures:
  - A lattice is a structure $(A, \vee, \wedge)$ that satisfies the lattice axioms.
  - A bounded lattice is a structure $(A, \vee, \wedge, \bot, \top)$ that satisfies the lattice axioms and the bounds axioms.
  - A (bounded) lattice is distributive if is additionally satisfies the distributivity axiom.

Comments III

- A Boolean algebra also can equivalently be described just in terms of the order: A partial order such that
  - any finite subset S has a least upper bound $\bigvee S$ and a greatest lower bound $\bigwedge S$ (so $\bot = \bigvee \emptyset$ and $a \vee b = \bigvee\{a, b\}$, similarly for $\top$ and $\wedge$),
  - it satisfies the distributivity axiom $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, and
  - every element a has a complement, i.e., there is another element b such that $a \wedge b = \bot$ and $a \vee b = \top$.

Important example 1: the two-element Boolean algebra

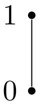
- The simplest nontrivial Boolean algebra is the two-element Boolean algebra $\mathbf{2} = (\{0,1\}, \vee, \wedge, \neg, 0, 1)$ with the usual Boolean truth-functions

$\vee$	0	1
0	0	1
1	1	1

$\wedge$	0	1
0	0	0
1	0	1

$\neg$	
0	1
1	0

- Notation: In set theory, $0 := \emptyset$ and $n := \{0, \ldots, n-1\}$. So $2 = \{0,1\}$.
- The order $\leq$ is given by $0 \leq 1$, and we depict it as



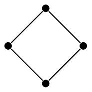
- In fact, every Boolean algebra is a subalgebra of a product of $\mathbf{2}$. So, in a sense, $\mathbf{2}$ generates all Boolean algebras! (But we do not show this here.)

Important example 2: the powerset

- If X is a set, then the power set $\mathcal{P}(X) = (\mathcal{P}(X), \cup, \cap, \cdot^c, \emptyset, X)$ is a Boolean algebra with the operations of union, intersection, and set-theoretic complement (for $a \subseteq X$, $a^c = X \setminus a$).
- The order is inclusion: For $a, b \subseteq X$, $a \leq b$ iff $a \cap b = a$ iff $a \subseteq b$.
- Cf. X is a set of possible worlds and $a \subseteq X$ are propositions.
- So $\mathbf{2}$ is simply the powerset Boolean algebra of a one-element set $\{\bullet\}$.
- The powerset of the empty set $\emptyset$ has one element. The powerset of a two-element set $\{\bullet, *\}$ has four elements:

$\mathcal{P}(\emptyset)$

$\mathcal{P}(\{\bullet\})$



$\mathcal{P}(\{\bullet, *\})$

Important example: the Lindenbaum–Tarski algebra

- Let $\mathcal{L}$ be the set of sentence of classical propositional logic formed by

$$p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \neg\varphi \mid \bot \mid \top$$

- Take your favorite proof system for classical propositional logic and write $\varphi \vdash \psi$ if ψ is derivable from φ.
- Define the equivalence relation $\varphi \equiv \psi$ by $\varphi \vdash \psi$ and $\psi \vdash \varphi$.
- Let $\mathcal{L}/\equiv := \{[\varphi] : \varphi \in \mathcal{L}\}$ be the quotient, with $[\varphi] := \{\psi : \varphi \equiv \psi\}$
- Define $[\varphi] \vee [\psi] := [\varphi \vee \psi]$, $[\varphi] \wedge [\psi] := [\varphi \wedge \psi]$, and $\neg[\varphi] := [\neg\varphi]$.
(Exercise: Check that this is well-defined.)
- Then $(\mathcal{L}/\equiv, \vee, \wedge, \neg, [\bot], [\top])$ is a Boolean algebra, called the Lindenbaum–Tarski algebra. (Cf. inferentialism.)

Boolean algebra homomorphism

Definition

Let $A = (A, \vee_A, \wedge_A, \neg_A, \bot_A, \top_A)$ and $B = (B, \vee_B, \wedge_B, \neg_B, \bot_B, \top_B)$ be two Boolean algebras. A Boolean algebra homomorphism is a function $h : A \to B$ such that, for all $a, b \in A$,

1. $h(a \vee_A b) = h(a) \vee_B h(b)$
2. $h(a \wedge_A b) = h(a) \wedge_B h(b)$
3. $h(\neg_A a) = \neg_B h(a)$
4. $h(\bot_A) = \bot_B$
5. $h(\top_A) = \top_B$

If h additionally is injective (resp., bijective), we call it a Boolean algebra embedding (resp., isomorphism). Two Boolean algebras are isomorphic if there is an isomorphism between them.

Comments

- Intuition: Translation from one language/set of propositions A to another B. Preservation conditions ensure compositionality: roughly, to translate a complex proposition in A, first translate its atoms and then build it up again in B.
- Boolean algebra homomorphism are order-preserving: If $a \leq_A b$, then $h(a) \leq_B h(b)$. (Exercise: prove this.)
- Not all conditions need to be required: For example, h already is a Boolean algebra homomorphism once it preserves $\neg, \wedge, \bot$. (Exercise: prove this.)

Outline

Overview of the section

- In this section, we define what Stone spaces are.
- We first start by defining topological spaces (and motivate the definition)
- We define further notions: when a space is Hausdorff, zero-dimensional, and compact.
- Then we define Stone spaces as compact, zero-dimensional Hausdorff topological spaces.
- Finally, we define continuous maps between topological spaces.

Topological spaces I

- When we hear of 'space', we naturally think of the three-dimensional space we live in, i.e., the Euclidean space $\mathbb{R}^3$ whose points $x = (x_1, x_2, x_3)$ are described by the values on the x-axis, the y-axis, and the z-axis.
- But there also are other spaces:
  - The surface of a sphere: its points are those (x_1, x_2, x_3) with $x_1^2 + x_2^2 + x_3^2 = 1$. But its geometry is different: for instance, the angles of a triangle add up to more than 180 degrees.
  - The spacetime that we live in according to general relativity: its points $x = (x_1, x_2, x_3, x_4)$ are four-dimensional—with three spatial and one temporal component—and its geometry is given by a metric tensor.
  - And there are even wilder spaces, in which it might not even make sense to speak of a 'geometry' (e.g., angles between lines), but only of 'spatial' properties (e.g., continuous paths from one point to another).

Topological spaces II

- Mathematicians came up with a general definition of a topological space that includes all these examples.
- It is very abstract, but we will see than it can recover all the relevant spatial concepts.

Definition

A topological space is a pair (X, τ) where X is a set and τ is a collection of subsets of X such that

1. $\emptyset$ and X are in τ
2. If $U, V \in \tau$, then $U \cap V \in \tau$
3. If $U_i \in \tau$ is a collection of sets indexed by a set I, then $\bigcup_{i \in I} U_i \in \tau$.

We also call τ a topology on X.

Topological spaces III

Definition (continued)

If (X, τ) is a topological space, we call the elements of X points. The elements of τ are called open sets (or opens). Their complements, i.e., sets of the form $C = X \setminus U$ for $U \in \tau$, are called closed sets. A subset $K \subseteq X$ that is both open and closed (i.e., $K \in \tau$ and $K^c \in \tau$) is called clopen.
We just speak of the topological space X if τ is clear from context. Then we write $\Omega(X)$ for the opens of X. The collection of closed (resp. clopen) subsets of X is denoted $\mathcal{C}(X)$ (resp. $\mathsf{Clp}(X)$).

Examples

1. The Euclidean topology on $\mathbb{R}^3$ is given by calling those sets $U \subseteq \mathbb{R}^3$ open that allow some 'wiggle-room', i.e.,
   - for all $x \in U$, there is $\epsilon > 0$ such that for all $x' \in \mathbb{R}^3$ with $d(x, x') < \epsilon$, we have $x' \in U$.

   Here the distance between two points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is, as usual,

   $$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

2. For a set X, the discrete topology on X is $\tau := \mathcal{P}(X)$.
3. For a set X, the indiscrete topology on X is $\tau := \{\emptyset, X\}$.

Topological concepts I

With just 'open set talk' we can recover many spatial concepts: Let (X, τ) be a topological space.

- Interior, closure, boundary. If $S \subseteq X$ is a subset, the interior $\mathsf{Int}(S)$ of S is the largest open set contained in S, and the closure $\mathsf{Cl}(S)$ of S is the smallest closed set containing S. The boundary ∂S of S is $\mathsf{Cl}(S) \setminus \mathsf{Int}(S)$.
- Neighborhood. A subset $S \subseteq X$ is a neighborhood of a point $x \in X$ if $x \in \mathsf{Int}(S)$. An open neighborhood of x is an open set that is a neighborhood of x. (If clear from context, we drop 'open'.)
- Dense. A subset $S \subseteq X$ is dense (in X) if for all points $x \in X$ and open neighborhoods U of x, there is a point $s \in S$ with $s \in U$. Equivalently, $\mathsf{Cl}(S) = X$.

Topological concepts II

- Convergence. A sequence $(x_n)_{n\in\mathbb{N}}$ of points in X converges to a point $x \in X$ if for all open neighborhoods U of x, there is $N \geq 0$ such that, for all $n \geq N$, we have $x_n \in U$. We also say that x is the limit of the sequence (x_n).
- Generated topology. Any collection $\mathcal{S}$ of subsets of a nonempty set Y generates a topology $\langle \mathcal{S} \rangle$ on Y: namely, the smallest topology on Y that contains all subsets in $\mathcal{S}$.
- Subbase. Given the topology τ on X, a collection $\mathcal{S}$ of subsets of X is called a subbase of τ if $\tau = \langle \mathcal{S} \rangle$. So the opens of τ are arbitrary unions of finite intersection of subbasic elements.

Topological concepts III

- Base. A base for the topology τ is a collection $\mathcal{S} \subseteq \tau$ such that for every point $x \in X$ and every open neighborhood U of x, there is $V \in \mathcal{S}$ such that $x \in V \subseteq U$. Equivalently, a base is a collection of open subsets of X such that every open set is a union of elements from the base.

Separation axioms

- There is an important classification of topological spaces according to which so-called separation axioms they satisfy.
- There are many such axioms and they all are of the form that two distinct points can be—in various senses—separated by the topology.

Definition

Let (X, τ) be a topological space.

1. X is T_0 (aka Kolmogorov) if, for all $x \neq y$ in X, there is an open $U \subseteq X$ such that U contains exactly one of x and y.
2. X is T_1 (aka Fréchet) if, for all $x \neq y$ in X, there is an open $U \subseteq X$ such that $x \in U$ and $y \notin U$.
3. X is T_2 (aka Hausdorff) if, for all $x \neq y$ in X, there are disjoint opens $U, V \subseteq X$ such that $x \in U$ and $y \in V$.

Compactness

Compactness formalizes the idea a space does not extend infinitely but has finite bounds.

Definition (Compactness)

Let X be a topological space. If $S \subseteq X$ is a subset, an open cover $\mathcal{U}$ of S is a collection of open sets such that $S \subseteq \bigcup_{U \in \mathcal{U}} U$. A subset $S \subseteq X$ is compact if every open cover $\mathcal{U}$ of S contains a finite subcover, i.e., there is a finite subset $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\mathcal{U}_0$ is an open cover of S. The space X is called compact if $S := X$ is compact.

(An equivalent definition: via finite intersection property.)

(Zero-) Dimensionality

- There is a purely topological way to determine the dimension of a space. The Euclidean space $\mathbb{R}^3$ then indeed gets dimension 3.
- (In fact, there are several ways to do this: small inductive dimension, large inductive dimension, and the Lebesgue covering dimension, see [Eng89, ch. 7].)
- Here we only need what it means for a space to have dimension zero. In that special case, the definition becomes the following.

Definition

A topological space (X, τ) is zero-dimensional iff X has a base consisting of clopen sets.

Stone spaces

Finally, we can define what a Stone space is.

Definition

A Stone space is a topological space that is compact, zero-dimensional and Hausdorff.

Examples:

- Every finite set with the discrete topology is a Stone space.

Important example: Cantor space

- Points: The set $2^{\mathbb{N}}$ of all binary sequences (i.e., functions $\mathbb{N} \to 2 = \{0, 1\}$)
- Topology: The topology τ generated by the cylinder sets: given a finite binary string $\sigma = \sigma(0), \ldots, \sigma(n-1)$, its cylinder set

$$[\sigma] := \{x \in 2^{\mathbb{N}} : \forall k \in \{0, \ldots, n-1\}.x(k) = \sigma_k\},$$

is the set of all binary sequences that have σ as initial segment.

Continuous functions

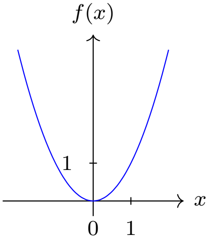
- Finally, we consider what it means to continuously map points of one space to another.
- Again, this has a swift abstract definition, which we then illustrate in more intuitive terms.

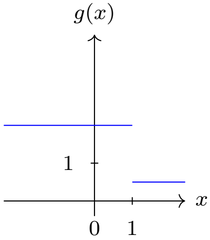
Definition

Let X and Y be topological spaces and $f : X \to Y$ a function. We say f is continuous if, for all open subsets V of Y, the preimage $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is an open subset of X.
A function $f : X \to Y$ is a homeomorphism (the topologists' name for isomorphism) if f is continuous and has a continuous inverse, i.e., f is a bijection and both f and f^{-1} are continuous. Two spaces are homeomorphic if there is a homeomorphism between them.

Continuous functions: intuition

- Maybe you have encountered continuous functions before but in the more specific epsilon–delta definition: It says $f : \mathbb{R} \to \mathbb{R}$ is continuous if
  - For every $x \in \mathbb{R}$ and every $\epsilon > 0$, there is $\delta > 0$ such that, for all $y \in \mathbb{R}$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.
- This captures the idea that, to draw the graph of the function, you do not have to lift your pen.
- This 'hands-on' definition (pun intended) is a special case of the abstract definition.





Outline

Overview of the section

- In this section, we finally get to the Stone duality theorem.
- It provides an exact correspondence between Boolean algebras on the one hand and Stone spaces on the other hand.
- In the first subsection, we state the theorem precisely (cf. the informal presentation in the first section).
- In second subsection, we consider (a sketch of) the proof.

Outline

From Boolean algebras to Stone spaces: Prime filters

- If we have a Boolean algebra A of propositions, we can build 'Ersatz' possible worlds as 'maximally consistent' sets F of propositions:

Definition

Let A be a Boolean algebra. A subset F of A is a filter if

- F is an upset, i.e, for all $a, b \in A$, if $a \in F$ and $a \leq b$, then $b \in F$.
- F is $\wedge$-closed, i.e., for $a, b \in F$ also $a \wedge b \in F$.
- F is nonempty (equivalently, $\top \in F$)

Moreover, F is proper if $F \neq A$ (equivalently $\bot \notin F$). A prime filter on A is a proper filter such that, for all $a, b \in A$, if $a \vee b \in F$, then either $a \in F$ or $b \in F$. We write $\mathsf{Spec}(A)$ or $\mathsf{PrFilt}(A)$ for the set of all prime filters on A.

Equivalent characterizations of prime filters

If A is a Boolean algebra and $F \subseteq A$ is a filter, the following are equivalent:

1. F is a prime filter.
2. The characteristic function $\chi_F : A \to \mathbf{2}$ (which maps a to 1 iff $a \in F$) is a Boolean algebra homomorphism.
3. F is an ultrafilter, i.e., F is proper and for any $a \in A$, either $a \in F$ or $\neg a \in F$.
4. F is a maximal filter, i.e., F is proper and for any proper filter G with $F \subseteq G$, we have $F = G$.

Thus, prime filters are like possible worlds/models of propositional logic:

- They compositionally decide for each proposition its truth-value.
- A maximal filter captures the idea that model is maximally consistent.

More comments on prime filters

- This equivalence fails if A is only a lattice and not a Boolean algebra.
- The order-dual notion of a filter is an ideal: a nonempty downset that is $\vee$-closed. Here we primarily work with filters, but some textbooks work with ideals.

From Boolean algebras to Stone spaces: Dual space

Proposition

Let A be a Boolean algebra. Equip $X := \mathsf{PrFilt}(A)$ with the topology generated by $\mathcal{S} := \{\widehat{a} : a \in A\}$, where

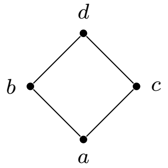
$$\widehat{a} := \{F \in \mathsf{PrFilt}(A) : a \in F\}.$$

Then X with this topology is a Stone space. Moreover, $\mathcal{S} = \mathsf{Clp}(X)$. We call $\mathsf{PrFilt}(A)$ the dual space of A.

- Open sets represent degrees of closeness/similarity: e.g., points in the open ball $B_{\epsilon/2}(x)$ are close to degree ϵ.
- Here, two prime filters are close to degree $a \in A$ if they agree on proposition a.

Example of dual space I

- Consider the four-element Boolean algebra A:



- What are its prime filters? To compute them by brute force, let's first list all its upsets:

$$\emptyset, \{\top\}, \{a, \top\}, \{b, \top\}, \{a, b, \top\}, \{\bot, a, b, \top\}.$$

- Which of those are filters? Not the empty set; not the second-last set (not $\wedge$-closed). But the remaining sets are filters.

Example of dual space II

- The remaining proper filters are $\{\top\}, \{a, \top\}, \{b, \top\}$.
- Which are prime? Not $\{\top\}$, since $a \vee b = \top$, but neither a nor b are in the filter.
- But the two filters

$$F_1 := \{a, \top\} \qquad\qquad F_2 := \{b, \top\}$$

are prime. They form the points of the dual space $X = \{F_1, F_2\}$.

Example of dual space III

- What is the topology of the space X? We in particular have the open sets:

$$\widehat{\bot} = \{F \in X : \bot \in F\} = \emptyset$$
$$\widehat{a} = \{F \in X : a \in F\} = \{F_1\}$$
$$\widehat{b} = \{F \in X : b \in F\} = \{F_2\}$$
$$\widehat{\top} = \{F \in X : \top \in F\} = \{F_1, F_2\}$$

- These already are all the subsets of the space X, so the topology is discrete. (later we will see that this is no coincidence).

From Stone spaces to Boolean algebras: Clopens I

- Informally, if we have a space X of possible worlds, then 'possible worlds semantics' propositions—aka intensions—are sets of possible worlds.
- But they are not just any subset: plausibly, a proposition/intension $a \subseteq X$ is a clopen set:
  - if x makes true a, there is a degree of similarity (i.e., an open set) such that all worlds y similar to x to this degree also make true a, and
  - if x does not make true a, there is a degree of similarity such that all worlds y similar to x to this degree also do not make true a.

From Stone spaces to Boolean algebras: Clopens II

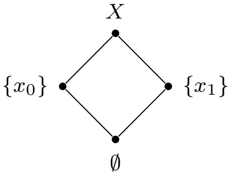
So the propositions/intensions are the clopen subsets of a given space X of possible worlds. They form a Boolean algebra:

Proposition

Let (X, τ) be a Stone space. Then $\mathsf{Clp}(X) = (\mathsf{Clp}(X), \cap, \cup, \cdot^c, \emptyset, X)$ is a Boolean algebra. We call $\mathsf{Clp}(X)$ the dual algebra of X.

Example of dual algebra

- Let $X = \{x_0, x_1\}$ be the two-element space with the discrete topology.
- Up to homeomorphism, this is the Stone space $\{F_0, F_1\}$ dual to the diamond Boolean algebra form the previous example.
- What is the Boolean algebra $\mathsf{Clp}(X)$?
- Let's first list all the possible subsets: $\emptyset, \{x_0\}, \{x_1\}, X$. Since the topology is discrete, all of them are clopen. They form the Boolean algebra



- We will see next, why this Boolean algebra is isomorphic to the diamond Boolean algebra we started out with.

Boolean algebras are isomorphic to their double-duals

- Assume A is a Boolean algebra of propositions.
- Given a proposition $a \in A$, consider its intension $\widehat{a} = \{F \in \mathsf{PrFilt}(A) : a \in F\}$ over the Ersatz worlds.
- We expect a to play the same role as $\widehat{a}$, i.e., the map $a \mapsto \widehat{a}$ should be an isomorphism. Indeed:

Proposition

Let A be a Boolean algebra. Then the following is a well-defined Boolean algebra homomorphism between A and its double-dual:

$$\begin{aligned} \widehat{\cdot} : A &\to \mathsf{Clp}(\mathsf{PrFilt}(A)) \\ a &\mapsto \widehat{a} = \{F \in \mathsf{PrFilt}(A) : a \in F\}. \end{aligned}$$

Spaces are isomorphic to their double-duals

- Assume X is a space of possible worlds.
- Is X homeomorphic to the space of Ersatz worlds built from the algebra of intensions over X?
- Each world x determines the set F_x of intensions that are true at x.
- This map $x \mapsto F_x$ turns out to be a homeomorphism!

Proposition

Let (X, τ) be a Stone space. The following is a well-defined homeomorphism:

$$\begin{aligned} \beta : X &\to \mathsf{PrFilt}(\mathsf{Clp}(X)) \\ x &\mapsto F_x = \{a \in \mathsf{Clp}(X) : x \in a\} \\ \text{the } x \text{ with } x \in a \text{ for all } a \in F &\mapsfrom F. \end{aligned}$$

Correspondence between morphisms

- Now we have the correspondence between the objects (Stone spaces and Boolean algebras).
- But we also want a correspondence between the morphisms (continuous maps and Boolean algebra homomorphisms).
- Again, we first see how to move in both directions,
- and then we check that these moves cancel each other out.

Translating morphisms I

From spaces to algebras:

- If we have a continuous function $f : X \to Y$, how to get a morphism between the dual Boolean algebras $\mathsf{Clp}(X)$ and $\mathsf{Clp}(Y)$?
- We could try mapping a clopen $a \subseteq X$ to the direct image $f[a] = \{f(x) : x \in a\} \subseteq Y$. But why would this be clopen and preserve the Boolean operations?
- A trick to remember: the preimage is generally much better behaved! So map $b \subseteq Y$ to $f^{-1}(b) = \{x \in X : f(x) \in b\} \subseteq X$.
- The next proposition show that this indeed is a Boolean algebra homomorphism.

Translating morphisms II

From algebras to spaces:

- If we have a Boolean algebra homomorphism $h : A \to B$, how to get a morphism between the dual Stone spaces $\mathsf{PrFilt}(A)$ and $\mathsf{PrFilt}(B)$?
- Again, we expect to go in the opposite direction: $\mathsf{PrFilt}(B) \to \mathsf{PrFilt}(A)$.
- If G is a prime filter on B, it is, in particular, a subset of B, so we can consider $h^{-1}(G) \subseteq A$, which indeed is a prime filter.
- Thus, we get the following translations.

Translating morphisms III

Proposition

1. *If $f : X \to Y$ is a continuous function between Stone spaces, then*

$$\mathsf{Clp}(f) : \mathsf{Clp}(Y) \to \mathsf{Clp}(X) \qquad B \mapsto f^{-1}(B)$$

is a Boolean algebra homomorphism.

2. *If $h : A \to B$ is a Boolean algebra homomorphism, then*

$$\mathsf{PrFilt}(h) : \mathsf{PrFilt}(B) \to \mathsf{PrFilt}(A) \qquad G \mapsto h^{-1}(G)$$

is a continuous function.

Translations cancel each other out I

- Finally, we check that also the translations of morphisms via PrFilt and Clp cancel each other out.
- We want that $f : X \to Y$ is 'isomorphic' to $\mathsf{PrFilt}(\mathsf{Clp}(f))$, but what should that mean?
- The functions f and $\mathsf{PrFilt}(\mathsf{Clp}(f))$ should behave in the same way under the isomorphisms that we already have for the objects, namely, $\beta_X : X \to \mathsf{PrFilt}(\mathsf{Clp}(X))$ and $\beta_Y : Y \to \mathsf{PrFilt}(\mathsf{Clp}(Y))$.
- Similarly, $h : A \to B$ and $\mathsf{Clp}(\mathsf{PrFilt}(h))$ should behave in the same way under the isomorphisms that we already have for the objects, namely, $\widehat{\cdot}_A : A \to \mathsf{Clp}(\mathsf{PrFilt}(A))$ and $\widehat{\cdot}_B : B \to \mathsf{Clp}(\mathsf{PrFilt}(B))$.
- This is made precise as follows.

Translations cancel each other out II

Proposition

The following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "\beta_X"] \arrow[d, "f"] & \mathsf{PrFilt}(\mathsf{Clp}(X)) \arrow[d, "\mathsf{PrFilt}(\mathsf{Clp}(f))"] \\ Y \arrow[r, "\beta_Y"] & \mathsf{PrFilt}(\mathsf{Clp}(Y)) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\widehat{\cdot}_A"] \arrow[d, "h"] & \mathsf{Clp}(\mathsf{PrFilt}(A)) \arrow[d, "\mathsf{Clp}(\mathsf{PrFilt}(h))"] \\ B \arrow[r, "\widehat{\cdot}_B"] & \mathsf{Clp}(\mathsf{PrFilt}(B)) \end{tikzcd}$$

Stone's theorem

- Stone's theorem is the conjunction of all the propositions that we have stated in this subsection.
- All the data in those propositions is category-theoretically captured by the notion of a dual equivalence:
- Thus, the swift category-theoretic formulation of Stone's theorem is:

Theorem (Stone duality)

The functors PrFilt *and* Clp *form a dual equivalence between, on the one side, the category* BA *of Boolean algebras with Boolean algebra homomorphisms and, on the other side, the category* Stone *of Stone spaces with continuous functions.*

Outline

Overview of the proof

- To prove Stone's theorem, we need to prove all propositions from the previous subsection.
- For reasons of time, we will not cover the whole proof in detail. Instead, we follow on the main ideas.
- But you can find the full proof in the linked lecture notes.
- These main ideas are already in the proof of the first proposition, i.e., that $\mathsf{PrFilt}(A)$ is a Stone space.
- The other propositions are either straightforward or use similar ideas.

The two main ideas of the proof

1. How to extend (if possible) subsets of a Boolean algebra into a filter and then into a prime filter.
2. Use this to prove compactness of $\mathsf{PrFilt}(A)$. This is done as follows:
   - Use the first step to prove the following version of the compactness theorem from logic:
   - A 'theory' $T \subseteq A$ (theory = set of propositions) has a model (= prime filter containing all propositions in T), as soon as all finite subsets of T have a model.
   - Derive compactness of $\mathsf{PrFilt}(A)$ from this.

Generating filters

Proposition

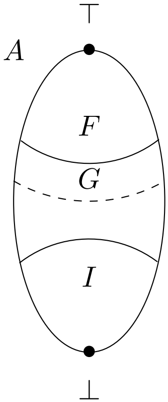
Let A be a Boolean algebra.

1. *For any subset $T \subseteq A$, there is a $\subseteq$-smallest filter F that contains T. It is called the filter generated by T and denoted $\langle T \rangle_{\text{filt}}$.*
2. *Concretely, this filter is given as*
$$\langle T \rangle_{\text{filt}} = \{a \in A : \text{ there is finite } T' \subseteq T \text{ such that } \bigwedge T' \leq a\}.$$
3. *If $F \subseteq A$ is a filter and we want to extend it by an element $b \in A$,*
$$\langle F \cup \{b\} \rangle_{\text{filt}} = \{a \in A : \text{ there is } f \in F \text{ such that } f \wedge b \leq a\}.$$

Prime filter extension

Theorem (Stone's Prime Filter Extension Theorem)

Let A be a Boolean algebra. If F is a filter and I and ideal in A such that $F \cap I = \emptyset$, then there is a prime filter G in A such that $F \subseteq G$ and $G \cap I = \emptyset$.



Proof that $\mathsf{PrFilt}(A)$ Stone space I

Let A be a Boolean algebra, let $X := \mathsf{PrFilt}(L)$ have topology generated by $\mathcal{S} := \{\widehat{a} : a \in L\}$ with $\widehat{a} := \{F \in X : a \in F\}$.

- Zero-dimensional: $\mathcal{S}$ is a base since closed under finite intersections: $\widehat{a_1} \cap \ldots \cap \widehat{a_n} = a_1 \widehat{\wedge \ldots \wedge} a_n$. Each $\widehat{a}$ is clopen: $\widehat{a}^c = \widehat{\neg a}$.
- Hausdorff: If $F \neq G$ in $\mathsf{PrFilt}(A)$, there is $a \in A$ with $a \in F$ and $a \notin G$ (or vice versa). So $F \in \widehat{a}$ and $G \in \widehat{\neg a}$ separate F and G.
- Compactness: We show that the compactness of X really is just the compactness theorem from logic. So we first formulate this logical compactness theorem (step 1), then show that it implies the topological compactness of X (step 2), and finally prove the logical compactness theorem (step 3).

Proof that $\mathsf{PrFilt}(A)$ Stone space II

1. A theory (in logic) is simply a set of sentences, so by an A-theory T we here mean a subset of A. Since models are prime filters, we say T has a model if there is a prime filter F of L such that $T \subseteq F$. Then the logical compactness theorem says:
   (∗) If T is an A-theory such that every finite subset T' of T has a model, then T has a model.
2. We show the compactness of X via the finite intersection formulation. Let $\mathcal{F}$ be a collection of closed subsets of X with the finite intersection property (FIP), and show that $\bigcap \mathcal{F} \neq \emptyset$. Since $\mathcal{S}$ is a base of clopens, assume $\mathcal{F} \subseteq \mathcal{S}$. So $\mathcal{F} = \{\widehat{a} : a \in T\}$ for some $T \subseteq A$. By FIP, every finite subset T' of T has a model: Write $T' = \{a_1, \ldots, a_n\}$, so there is some $F \in \widehat{a_1} \cap \ldots \cap \widehat{a_n}$, so $T' \subseteq F$, as needed. By (∗), T has a model F, so $F \in \bigcap \mathcal{F}$, as needed.

Proof that $\mathsf{PrFilt}(A)$ Stone space III

3. We prove $(*)$. Let $T \subseteq A$ such that every finite subset has a model. To show that T has a model, we need to extend T to a prime filter.
   First, let F be the filter generated by T, so
   $F = \{a \in A : \exists T' \subseteq T. \bigwedge T' \leq a\}$.
   Second, for $I := \{\bot\}$, we have $F \cap I = \emptyset$; otherwise $\bot \in F$, so there is a finite $T' \subseteq T$ with $\bigwedge T' \leq \bot$, but then T' cannot have a model. So Stone's Prime Filter Extension Theorem applies and yields a prime filter G such that $T \subseteq F \subseteq G$, as needed.

Outline

Overview of the section

- There are a plethora of applications of Stone duality and generalizations thereof. See, e.g., Gehrke and Gool [GG23, ch. 4–8].
- Many applications are in theoretical computer science (e.g., programming semantics or automata theory), we consider one in modal logic/philosophy.
- We add a less understood but important concept—here, necessity—to our logical concepts, and we get a better understanding of it by studying its behavior on the dual side of spaces.

(Re-) discovering Kripke semantics

1. We see that the necessity operator is, like the Boolean operators, a function on the set of all propositions. However, unlike the Boolean operators, it is not clear what its truth-conditions are.
2. We apply Stone duality, to move this question from propositions to possible worlds, and find an answer there.
3. We see that this is exactly the famous answer Kripke gave: the Kripke semantics for the necessity operator.

See, e.g., [JT51; JT52; Gol00].

The 'necessity' operator

- Assume we have a Boolean algebra A of propositions.
- In philosophical context, the Boolean connectives/operators $\vee, \wedge, \neg, \bot, \top$ are often not enough.
- We also would like to say that some propositions are necessarily true, while others only contingently. (Other operators: Knowledge, belief, eternity, essence, etc.)
- So, if φ is a sentence expressing proposition a, we also consider the sentence 'Necessarily, φ' (written $\Box\varphi$), which expresses the proposition that it is necessarily the case that a.
- Just like Boolean connectives become operators on the Boolean algebra of propositions, so does $\Box$:

$$\Box : A \to A, \quad a \mapsto \Box a.$$

The search for truth-conditions of 'necessity'

- Before the advent of Kripke semantics, philosophers—like us now—struggled to understand the necessity connective.
- The Boolean connectives they could describe by their truth-conditions: $\varphi \vee \psi$ is true (in some situation) iff either φ is true (in that situation) or ψ is true (in that situation), etc.
- However, for 'necessity', they could only provide some plausible reasoning principles, e.g.,
  1. It is necessary that φ and ψ if and only if it is necessary that φ and it is necessary that ψ.
  2. If φ is a logical truth, then 'it is necessary that φ' also is a logical truth.
- Cf. only having inference rules for disjunction but no truth-tables for $\vee$.

Duality to the rescue?

- Can we use duality theory to find truth conditions by translating to the spatial side?
- What we know so far: $\Box$ is some (unknown) function $A \to A$ such that
  1. For all $a, b \in A$: $\Box(a \wedge b) = \Box a \wedge \Box b$.
  2. $\Box\top = \top$.
- How to translate $\Box : A \to A$ to the dual space $\mathsf{PrFilt}(A)$?
- Problem: $\Box$ is not a Boolean algebra homomorphism (only preserves $\wedge$ and $\top$), so we cannot directly take $\mathsf{PrFilt}(\Box) : \mathsf{PrFilt}(A) \to \mathsf{PrFilt}(A)$ mapping F to $\Box^{-1}(F)$.

Finding the dual

- The preservation of $\top$ and $\wedge$ only yields that $\Box^{-1}(F)$ is a filter, not necessarily a prime filter.
- The next best thing is to consider the *set* of all prime filters that extend this filter:

$$f(F) := \{G \in \mathsf{PrFilt}(A) : G \supseteq \Box^{-1}(F)\}.$$

- We say $f : \mathsf{PrFilt}(A) \to \mathsf{PrFilt}(A)$ is a multifunction (mapping points to sets of points).
- It can equivalently be described as a relation (its graph):

$$F \; R \; G \Leftrightarrow G \supseteq \Box^{-1}(F) \Leftrightarrow \forall a \in A : \Box a \in F \Rightarrow a \in G.$$

Finding the truth-conditions

- So our function $\Box : A \to A$ on the algebraic side becomes, when translated to the spatial side, the relation R.
- But this doesn't yet give us truth-conditions for the necessity operator.
- For $F \in \mathsf{PrFilt}(L)$, we need to complete $\Box a \in F \Leftrightarrow$???.
- By definition of R we get a necessary condition: If $\Box a \in F$, then, if $F \; R \; G$, we have $\forall a \in L : \Box a \in F \Rightarrow a \in G$, so, in particular, $a \in G$.
- This also is sufficient:

Theorem

In the above notation, we have for all $a \in A$ and prime filters $F \in \mathsf{PrFilt}(A)$

$$\Box a \in F \Leftrightarrow \forall G \in \mathsf{PrFilt}(A) : F \; R \; G \Rightarrow a \in G.$$

Rediscovering Kripke semantics

- The truth-conditions we thus found, are exactly those of Kripke semantics:
  - $\Box a$ is true at a possible world F iff for all R-accessible worlds G, we have that a is true at G.

Proof.

($\Rightarrow$) This was the necessary condition which we already saw to hold.
($\Leftarrow$) Contrapositively, assume $\Box a \notin F$. We want to find a prime filter $G \in \mathsf{PrFilt}(A)$ with $F\ R\ G$ but $a \notin G$. So turn to the Prime Filter Extension Theorem. We already noted that $\Box^{-1}(F)$ is a filter on A and, by assumption, $a \notin \Box^{-1}(F)$, hence, for the ideal $I := {\downarrow} a$, we have $\Box^{-1}(F) \cap I = \emptyset$. By the Extension Theorem, there is a prime filter G on A such that $G \supseteq \Box^{-1}(F)$ and $G \cap I = \emptyset$. The former means, by definition, that $F\ R\ G$, and the latter implies $a \notin G$, as needed. $\Box$

Outlook

- We have come to the end of the course!
- Congratulations on making it this far!
- Now you know how to translate between Boolean algebras and Stone space, and why they are two sides of the same coin.
- If you want to continue, you could consider, e.g.:
  1. The duality between bounded distributive lattices and spectral/Priestley spaces [GG24]
  2. The duality between Heyting algebras and Esakia spaces [GG24]
  3. The applications to programming semantics [Abr91] and automata theory [GG24]
  4. Extensions to first-order logic [Mak87; AF13]

Thank you!

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