Example Exam

Duality Theory

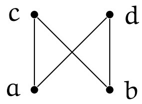
Disclaimer: This example exam aims to be representative of the final exam—though, exam difficulty is always somewhat subjective (some topics might feel more familiar or easier to you than others, etc.). Note, however, that this practice exam is *only based on material that we have covered in class so far*; the final exam can contain material up to the last lecture. Moreover, the practice exam might also have a different structure than the final exam (more or less exercises, different distribution of points, different topics, etc.).

The exam will be open-book, so you can access the lecture notes. But this should only *remove the need for memorizing, not the need for knowing the material*. There will not be enough time to read up on every exercise before solving it, but the notes will help, e.g., to quickly look up a formal definition.

When doing this practice exam, it is recommended to take *two undisturbed hours* for yourself, to really mimic the exam situation (which will last 120 minutes). This is because it is very different to solving exercises at home during a whole week. After having done your own attempt, you can of course discuss this example exam with others.

The exercises are roughly increasing in difficulty.

Exercise 1 (25 points) Consider the following partial order:

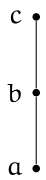


Tick your answer (no proof needed):

The set $\{a, b\}$ has an upper bound	☐ Yes	☐ No
The set $\{a, b\}$ has a least upper bound.	☐ Yes	☐ No
a is a minimal element.	☐ Yes	☐ No
a is the least element.	☐ Yes	☐ No
$\{a, c\}$ is an upset.	☐ Yes	☐ No

—please turn the page—

Exercise 2 (25 points) Consider the following partial order:



Is this a Boolean algebra? If so, provide a proof; if not, show why not.

Exercise 3 (25 points) Let 2^ω be Cantor space. Let X be a Hausdorff space and let $\mathcal{C} = \{U_n : n \in \omega\}$ be a countable collection of clopen subsets of X that forms a base. Consider the function $f : X \to 2^\omega$ that maps $x \in X$ to the binary sequence $f(x) = y \in 2^\omega$ defined by

$$y_n := \begin{cases} 0 & x \notin U_n \\ 1 & x \in U_n \end{cases}$$

1. Prove that f is an embedding, i.e., an injective continuous function such that, for any open $U \subseteq X$, there is an open $V \subseteq 2^\omega$ such that $f[U] = f[X] \cap V$.

2. Use 1 to show that the set of irrational numbers $\mathbb{I} \subseteq \mathbb{R}$ equipped with the subspace topology can be embedded into the Cantor space.

Exercise 4 (25 points) Let X be a nonempty set. Let $\mathcal{P}(X)$ be the Boolean algebra of subsets of X. For $x \in X$, define

$$F_x := \{A \in \mathcal{P}(X) : x \in A\}$$

1. Prove that, for each $x \in X$, the set F_x is a prime filter on $\mathcal{P}(X)$.

2. Assume that X is finite. Prove that, if F is a prime filter on $\mathcal{P}(X)$, then there is $x \in X$ such that $F_x = F$.

3. Now assume that X is infinite. Consider the collection F of those subsets A of X whose complement is finite. Show that F is a proper filter on $\mathcal{P}(X)$ and that no prime filter G extending F can be of the form F_x for some $x \in X$.

   (Since proper filters on Boolean algebras can always be extended to prime filters, this shows that 2 indeed fails if X is infinite.)