Duality Theory
Connecting Logic, Algebra, and Topology

A Reader

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Comments welcome!

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Preface

This is the reader for the course “Duality Theory: Connecting Logic, Algebra, and Topology” given during the winter semester 2023/24 at *LMU Munich* as part of the *Master in Logic and Philosophy of Science*. The reader is written as the course progresses. A website (or rather git repository) with all the course material is found at

> `https://github.com/LevinHornischer/DualityTheory`.

Comments I’m happy about any comments: spotting typos, finding mistakes, pointing out confusing parts, or simply questions triggered by the material. Just send an informal email to `Levin.Hornischer@lmu.de`.

Course description and objectives This course is an introduction to duality theory, which is an exciting area of logic and neighboring subjects like math and computer science. The fundamental theorem is Stone’s duality theorem stating that certain algebras (Boolean algebras) are in a precise sense equivalent to certain topological spaces (totally disconnected compact Hausdorff spaces). This has been extended in many ways. The underlying idea is that the two seemingly different perspectives—the algebraic one and the spatial one—are really two sides of the same coin:

- formulas/propositions vs. models/possible worlds,
- open sets of a space vs. points of the space,
- properties of a computational process vs. denotation of the computational process.

In terms of content, the focus of the course will be to introduce the mathematical theory. In terms of skills, the aim is to learn how to apply the tools of duality theory. We will illustrate this with applications that make use of dualities by combining the often opposing advantages of the two perspectives.

Prerequisites An introductory course in logic and some familiarity with mathematics (ideally, but not necessarily, having seen elementary concepts of topology and algebra), including the basics of writing mathematical proofs.

Apart from that, the course can be taken independently. But it also makes sense to take it as a follow-up course of the course "Philosophical Logic", which I thought in the summer semester 2023. In that course, I stressed two different approaches to giving semantics to various logics: the algebraic approach and the state-based approach. We've seen that these semantic approaches are often equivalent, and this is a special case of the more general phenomenon of duality.

Contents We start with an informal chapter describing the key idea of duality. The rest of the course is about developing this key idea precisely. For this, we follow the recent textbook Gehrke and van Gool 2023. We first precisely define the algebraic structure (lattices) and then topological structures (topological spaces), and we finally prove the duality result. The remainder of the course is about deepening this result and applying it in logic and computer science.

Layout These notes are informal and partially still under construction. For example, there are margin notes to convey more casual comments that you'd rather find in a lecture but usually not in a book. Todo notes indicate, well, that something needs to be done. References are found at the end.

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Study material The main textbook that we use is by Gehrke and van Gool (2023). And informal introduction to duality is provided by Gehrke (2009). Some further textbooks include:

- R. Balbes and P. Dwinger (1975). *Distributive lattices*. University of Missouri Press
- B. A. Davey and H. A. Pristley (2002). *Introduction to Lattices and Order*. 2nd ed. Cambridge: Cambridge University Press
- S. Vickers (1989). *Topology via Logic*. Cambridge: Cambridge University Press
- S. Givant and P. Halmos (2008). *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. New York: Springer-Verlag
- S. Givant (2014). Ed. by D. theories for Boolean algebras with operators. Springer

- G. Grätzer (2011). *Lattice Theory: Foundation*. Birkhäuser
- G. Grätzer (2003). *General Lattice Theory*. 2nd ed. Birkhäuser

Research monographs on duality theory are

- P. T. Johnstone (1982). *Stone Spaces*. Cambrdige studies in advanced mathematics 3. Cambridge: Cambridge University Press
- G. Gierz et al. (2003). *Continuous Lattices and Domains*. Cambridge: Cambridge University Press
- M. Dickmann et al. (2019). *Spectral Spaces*. New Mathematical Monographs. Cambridge University Press. DOI: 10.1017/9781316543870
- J. Goubault-Larrecq (2013). *Non-Hausdorff Topology and Domain Theory*. Cambridge University Press
- J. Picado and A. Pultr (2012). *Frames and Locales*. Birkhäuser
- S. Abramsky and A. Jung (1994). "Domain Theory." In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky et al. Corrected and expanded version available at http://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf (last checked 24 January 2018). Oxford: Oxford University Press
- E. Orłowska et al. (2015). *Dualities for Structures of Applied Logic*. Studies in Logic 56. College Publications

Notation Throughout, 'iff' abbreviates 'if and only if'.

1 Introduction: the key idea of duality

Duality theory is a mathematical theory relating algebraic structures to geometric or spatial structures. It is a formal mathematical theory; but underlying it, is a deep philosophical idea. In this chapter, we describe this philosophical story—the key idea of duality—before developing the mathematical theory and its applications in the later chapters.

For other expositions of the philosophical idea behind duality, see, e.g., Abramsky (1991), Gehrke (2009), and Vickers (1989).

Advice on how to read this chapter. Duality theory can be confusing when one first hears about it. One has to keep track of many moving parts, going in different directions, making sure they all fit together. At least to me, reminding myself of the philosophical story helps: it provides the 'rhyme and reason' to the mathematics. So whenever you feel lost in the midst of the technical detail, you can come back to this philosophical story. It is a powerful and potentially unfamiliar idea, so give it some time to sink in and go through this conceptual motivation over and over again. Also, as you progress to the later, more technical chapters, be sure to come back to this introduction chapter to see how the intuitive ideas here are developed formally.

To use the words of Abramsky (2023).

Duality theory can be quite abstract. The advantage of this is that it makes duality ubiquitous and widely applicable. But a disadvantage is that this makes it less accessible. So before attempting any general definition of duality, let us consider several examples (section 1.1). From those we can generalize an informal characterization of duality (section 1.2). This then hints at how duality theory is formalized mathematically and how it can be applied. Finally, in section 1.3 we list some exercises.

1.1 Intuitive examples of duality

We present several examples of duality. We do so at a very informal and intuitive level, and we do not at all aim to be philosophically careful or mathematically precise. In fact, think of it as an *exercise* to revisit these examples once you know more about the formal development of duality theory—and see what more precise analysis you can provide.

I think this is a philosophically very fruitful exercise—or, better, research project. In particular, this makes for an excellent essay topic.

1.1.1 Metaphysics: Properties vs objects

When we perceive and reason about the world, we naturally think in terms of there being various objects that have—or do not have—various properties. Objects are, for example, my laptop, the Eiffel Tower, or the Moon. Properties are, for example, being red, being higher than $300\mathrm{m}$, or being made of cheese. (We consider here only unary properties: i.e., those that apply to a single object, but not to multiple objects, like being taller than.) Let us write $\mathcal{O}$ for the set of all objects and $\mathcal{P}$ for the set of all properties. Crucially, observe that there is a certain dependency between $\mathcal{O}$ and $\mathcal{P}$:

($\mathcal{O} \to \overline{\mathcal{P}}$) Each object $x \in \mathcal{O}$ determines a set of properties $F_x \subseteq \mathcal{P}$ consisting of precisely those properties that x has.

Philosophers also call F_x the role *of the individual x (McMichael 1983, p. 57).*

(The bar in '$\overline{\mathcal{P}}$' indicates that we assign to each x a *set* of elements in $\mathcal{P}$ rather than a single element of $\mathcal{P}$.) So we might wonder whether we can also go in the opposite direction ($\overline{\mathcal{P}} \to \mathcal{O}$)? Does a subset F of properties also determine an object, i.e., the unique object that has exactly the properties in F? Actually, no: some sets of properties might not be satisfied by any object (e.g., $F = \{$*being exactly* $300\mathrm{m}$ *high*, *being exactly* $200\mathrm{m}$ *high*$\}$) or by more than one (e.g., $F = \{$*being exactly* $300\mathrm{m}$ *high*$\}$).

Philosophers know phrases of the form 'The F' (referring to the unique object satisfying F) as definite description. For their important role in philosophy, see e.g. Ludlow (2022).

But let us not give up too early. After all, the set F_x is not just *any* set of properties, but it has some nice features which we collect now. (And the hope is that if F is a set of properties with these nice features, that then it determines a unique object.)

1. Assume $a, b \in \mathcal{P}$ are two properties such that having a implies having b; we abbreviate this as $a \leqslant b$. For example,

$$a = \textit{being higher than } 300\mathrm{m} \leqslant \textit{being higher than } 200\mathrm{m} = b.$$

   So if our object x has property a, then it also has property b, i.e., if $a \in F_x$, then $b \in F_x$. We may express this as: F_x is closed under implication.

Later we will say F_x is an upset*. This sounds funny now, but by the end of the course, you will have said this so often that you won't even notice.*

2. Assume $a, b \in \mathcal{P}$ are two properties. Note that then there is another property: namely, the property of having both property a and property b. We denote this property $a \wedge b$. So $a \wedge b$ is again in $\mathcal{P}$ and we have $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$. Moreover, if our object x has property a and it has property b, then it has property $a \wedge b$, i.e., if $a, b \in F_x$, then $a \wedge b \in F_x$. We may express this as: F_x is closed under conjunction.

3. Similarly, if $a, b \in \mathcal{P}$ are two properties, there also is the property of having either property a or property b (or both). We denote this property $a \vee b$. So $a \vee b$ is again in $\mathcal{P}$ and we have $a \leqslant a \vee b$ and $b \leqslant a \vee b$. Moreover, if our object x has property $a \vee b$, then either it has property a or it has property b, i.e., if $a \vee b \in F_x$, then either $a \in F_x$ or $b \in F_x$. Later, we express this as F_x being prime.

I will always read 'either A or B' as inclusive-or (either only A is the case, or only B is the case, or both A and B are the case)

Cf. a number $p > 1$ is prime iff (that is Euclid's lemma), for all numbers a and b, if $a \times b$ is divided by p, then either a is divided by p or b is divided by p).

4. Note that $\mathcal{P}$ also contains the trivial property like being identical to oneself. We denote this property $\top$. In particular, our object x has it, i.e., $\top \in F_x$.

5. Similarly, note that $\mathcal{P}$ also contains the inconsistent property like not being identical to oneself. We denote this property $\perp$. In particular, our object x does not have it, i.e., $\perp \notin F_x$.

Now, we can ask our question again: If F is a set of properties with these features, does *it*—as opposed to any arbitrary set of properties—determine a unique object? In other words, is there exactly one object that has all the properties in F? It might be an attractive metaphysical (or, better, ontological) principle to answer yes and hold that:

Or is the list (1)–(5) not complete because we should also add a principle concerning negation: you can think about this in exercise 1.b.

($\overline{\mathcal{P}} \to \mathcal{O}$) Each set of properties $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines an object $x \in \mathcal{O}$, namely, the unique object having exactly the properties in F.

The uniqueness part is close to Leibniz's principle about the identity of indiscernibles: if two objects x and x' have exactly the properties in F, they are indiscernible, and hence are identical according to Leibniz. The existence part amounts to a certain *ontological completeness*: that for every consistent description F of an object, there in fact is a (possible) object that has these properties. For this, we should consider $\mathcal{O}$ to contain not only the objects in our world, but all possible objects. After all, the actual world need not be ontologically complete: F might consistently describe a unicorn, even if this does not exist in the actual world.

Actually, I don't know if a principle like this is considered in metaphysics: if you do, please let me know :-) Also see exercise 1.c asking for a comparison to formal concept analysis.

We will see that this bidirectional determination ($\mathcal{O} \to \overline{\mathcal{P}}$) and ($\overline{\mathcal{P}} \to \mathcal{O}$) is a hallmark of duality, here between objects and properties. We might also speak of mutual dependency, supervenience, or necessitation.

Moreover, we started our considerations from objects and considered their ontology; but we could also start from properties and wonder about their ontology. The analog of Leibniz's principle would be the extensionality principle: two properties a and b are identical if they apply to exactly

Cf. the extensionality principle in set theory which says that two sets are identical iff they have the same elements.

the same possible objects (i.e., for all $x \in \mathcal{O}$, x has a iff x has b). Each property a determines a set of objects: namely, the set of those objects that have property a. This is known as the *extension* of the property. Analogously to before, we might also ask if every set of objects determines a property: namely, the property determined by having this set of objects as extension. Prima facie one would think that this should be the case, but we will see that duality provides a different answer: only some—and not all—sets of objects determine a property.

Since we talk about all possible objects, not just the actual ones, some philosophers might rather call this the intension of the property, as it involves not just the actual world, but also objects from other possible worlds.

1.1.2 Semantics: Propositions vs possible worlds

The central question of philosophy of language is: What is the meaning of sentences? The meaning of a sentence is also called the *proposition* that the sentence expresses. The standard answer to this question, as far as there is one, is possible worlds semantics: The meaning of a sentence (i.e., the proposition it expresses) is the set of possible worlds in which the sentence is true. Here, a possible world is a consistent and complete description of how our world could have been. One example is the possible world which is just like our world but where the Eiffel Tower is $400m$ high. So the proposition a expressed by the sentence 'The Eiffel Tower is $330m$ high' contains the actual world x_0 (i.e., $x_0 \in a$) but not the just described possible world x_1 (i.e., $x_1 \notin a$). Some common notation for the phrase 'world x makes true proposition a' is $x \vDash a$; so possible world semantics analyses $\vDash$ as elementhood $\in$.

There is much debate in philosophy what the set $\mathcal{W}$ of possible worlds is (Menzel 2021) and what the set $\mathcal{P}$ of propositions is (McGrath and Frank 2023). Both are taken to exist in their own right and be important objects of study. But their nature is disputed. For example, is it really the case, as possible world semantics claims, that propositions are just sets of worlds ('worlds first, propositions later')? Or is it rather that worlds are maximally consistent sets of propositions ('propositions first, worlds later')? The latter goes by the name 'ersatzism' since full-blown possible worlds are substituted by something constructed out of linguistic entities—and 'Ersatz' is German for substitute.

We won't enter this debate here. Instead, we observe again that there is a bidirectional determination between worlds and propositions. To start, a plausible principle to hold about worlds and propositions is the following. It is satisfied by possible worlds semantics, and, in fact, arguably its characteristic feature.

World individuation Possible worlds are individuated by the propositions they make true: if two possible worlds x and y make true exactly the same propositions (i.e., for every proposition a, we have $x \vDash a$ iff $y \vDash a$), then $x = y$.

Cf. Leibniz's above principle about the identity of indiscernibles.

Proposition individuation Propositions are individuated by the possible worlds at which they are true: if two propositions a and b are true at exactly the same possible worlds (i.e., for every possible world x, we have $x \vDash a$ iff $x \vDash b$), then $a = b$.

A hyperintensional account of propositions would contest this; see Berto and Nolan (2021).

And there is more. Just like properties, also the set of propositions has logical structure: If a and b are propositions, there also are the propositions $a \wedge b$ (conjunction), $a \vee b$ (disjunction), $\neg a$ (negation), $\top$ (logical truth), and $\bot$ (logical falsity). With this we can also express implications between propositions: proposition a implies proposition b, written $a \leqslant b$, precisely if $a \wedge b = a$. The proposition expressed by 'I am in Munich' implies the proposition expressed by 'I am in Germany' because the sentence 'I am in Munich and I am in Germany' is equivalent to the sentence 'I am in Munich', i.e., they express identical propositions.

Thus, given a possible world $x \in \mathcal{W}$, we can again consider the set of propositions $F_x \subseteq \mathcal{P}$ that are true in x (i.e., $F_x = \{a \in \mathcal{P} : x \vDash a\}$). And F_x again satisfies the features (1)–(5) above: If $a \in F_x$, i.e., $x \vDash a$, and a implies b, i.e., $a \leqslant b$, then $x \vDash b$, i.e., $b \in F_x$. If $a, b \in F_x$, then x makes true both a and b, so $a \wedge b \in F_x$. As an exercise, go through the other cases as well.

Another plausible principle to hold about worlds and propositions is, again, that

Metaphysical completeness Each set of propositions $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines a possible world $x \in \mathcal{W}$, namely, the unique possible world making true exactly the propositions in F.

Ersatzism, for example, endorses this principle; let us see why. We will later formally show that a set of propositions F satisfying (1)–(5) is maximally consistent: one cannot add a single more proposition to F without making it inconsistent (i.e., making it contain $\bot$). Ersatzism not only claims that then there is a world x which makes true exactly the propositions in F, it even identifies this world x with F. The metaphysic completeness claim only follows along with the existence claim, and the uniqueness of x follows from the world individuation principle above.

This is assuming that the set of propositions forms what is known as a Boolean algebra.

In other words, there is an exact match between possible worlds and sets of propositions satisfying (1)–(5). Formally, we say there is a bijective correspondence between the set $\mathcal{W}$ of possible worlds and the set $\overline{\mathcal{P}}$ of sets of propositions satisfying (1)–(5). (To anticipate terminology, these sets $F \in \overline{\mathcal{P}}$ will be called *prime filters* and $\overline{\mathcal{P}}$ will be called the *spectrum* of the algebra of propositions.)

$$\begin{aligned} \mathcal{W} &\leftrightarrows \overline{\mathcal{P}} \\ x &\mapsto F_x = \{a \in \mathcal{P} : x \vDash a\} \\ \text{the } x \text{ making true exactly the } a \in F &\leftarrow F \end{aligned}$$

Let us verify that this really is a bijection: We have already checked that the function $f : \mathcal{W} \to \overline{\mathcal{P}}$ mapping x to F_x is well-defined. It is injective by the world individuation principle: if $x \neq y$, then there is a proposition a with $x \vDash a$ and $y \nvDash a$ (or vice versa), so $a \in F_x$ and $a \notin F_y$ (or vice versa), so $F_x \neq F_y$. It is surjective by metaphysical completeness: Given $F \in \overline{\mathcal{P}}$, let x be the unique world in $\mathcal{W}$ making true exactly the propositions in F. Then $F = F_x$ because: $a \in F$ iff $x \vDash a$ iff $a \in F_x$.

A function $f : X \to Y$ is injective if $x \neq y$ implies $f(x) \neq f(y)$, it is surjective if for every $y \in Y$ there is $x \in X$ with $f(x) = y$, and it is bijective if it is both injective and surjective.

So far, we have looked at the relation between full-blown metaphysical worlds (the elements of $\mathcal{W}$) and their ersatz constructions as sets of propositions (the elements of $\overline{\mathcal{P}}$). But what about the other side: How do full-blown propositions (the elements of $\mathcal{P}$) relate to sets of worlds, i.e., their counterparts propagated by possible worlds semantics?

Every proposition $a \in \mathcal{P}$ determines the set of worlds $[\![a]\!] := \{x \in \mathcal{W} : x \vDash a\}$ where a is true. This is also known as the *truthset* of a. And we might again wonder whether we can also go in the opposite direction: whether every set of worlds also determines a proposition? This issue is actually not too much discussed in the philosophy of a language, and one often at least talks as if this is true. So let's see where this takes us. Let us write $\overline{\mathcal{W}}$ for the sets of worlds that determine propositions and $2^{\mathcal{W}}$ for the set of all sets of worlds. So our assumption for now is that $\overline{\mathcal{W}} = 2^{\mathcal{W}}$. Analogous to the previous case, we want to know if the function

If X is a set, the powerset of X is the set of all subsets of X and it is denoted 2^X or $\mathcal{P}(X)$.

$$\begin{aligned} [\![\cdot]\!] &: \mathcal{P} \to 2^{\mathcal{W}} \\ a &\mapsto [\![a]\!] = \{x \in \mathcal{W} : x \vDash a\} \end{aligned}$$

is a bijection. We are off to a good start: The function is injective by the proposition individuation principle: if $a \neq b$, there is a world x with $x \vDash a$ and $x \nvDash b$ (or vice versa), so $[\![a]\!] \neq [\![b]\!]$. In fact, it also preserves the logical structure: $[\![a \wedge b]\!] = [\![a]\!] \cap [\![b]\!]$, $[\![\bot]\!] = \emptyset$, etc. (Later we formalize this as $[\![\cdot]\!]$ being a Boolean algebra homomorphism.) However, the issue is surjectivity. (Above, this also required another assumption: metaphysical completeness.)

Here is one argument why $[\![\cdot]\!]$ is not surjective. Plausibly, since propositions are the meanings of sentences, every proposition is expressed by some sentence. But since there are only countably many sentences (they are generated by a 'finitistic' grammar), there hence only are countably many propositions. However, since there plausibly are infinitely many possible worlds (be it countably or uncountably many), the powerset $2^{\mathcal{W}}$ of $\mathcal{W}$ is uncountable. So $\mathcal{P}$ and $2^{\mathcal{W}}$ have different cardinalities, which means there cannot be a bijection between, hence the already injective function $[\![\cdot]\!]$ cannot be surjective.

That is Cantor's diagonal argument.

So actually not any set of worlds determines a proposition, i.e., $\overline{\mathcal{W}}$ is a proper subset of $2^{\mathcal{W}}$. The ingenious insight of Stone, who discovered the Stone duality, was to realize how to precisely describe this special subset $\overline{\mathcal{W}}$ of $2^{\mathcal{W}}$. The key idea is to realize that there is some additional structure on the set of worlds $\mathcal{W}$ that we have not seen so far: a topology. But this is something that needs more introduction, and we do this properly in the formal chapters.

Also see exercise 1.d.

So we have a duality between worlds and propositions: even if we do not endorse a particular view about one side—like possible worlds semantics or ersatzism—, the duality still describes a bidirectional determination between the two. So accepting principles on one side translates to the other side, where we can use a very different set of intuitions to test the principles.

1.1.3 Logic: models vs formulas

Logic can be done both syntactically (aka proof-theoretically) or semantically (aka model-theoretically). The completeness theorem shows that the two approaches, which are very different in spirit, actually are equivalent. This also is a form of duality. Let's explore this concretely.

Consider the language of classical propositional logic: sentences are formed from atomic sentences $p_0, p_1, \dots$ using the connectives $\wedge, \vee, \neg$ and the constants $\bot$ and $\top$. And consider a proof-system for classical logic: for example a Hilbert system, a natural deduction system, or a sequence calculus for classical logic. It consists of various axioms and rules to define the relation $\Gamma \vdash \varphi$, i.e., when the sentence φ is derivable in the proof-system S using as axioms the sentences in the set Γ. This is the syntactic description of the logic.

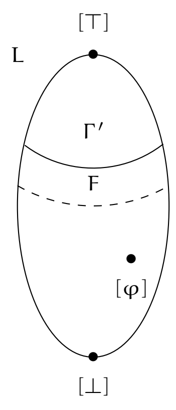
The model-theoretic description of the logic defines the relation $\Gamma \vDash \varphi$, i.e., that the sentence φ is a logical consequence of the sentences in Γ. This is done as follows. A valuation is a function $\nu : \{p_0, p_1, \ldots\} \to \{0, 1\}$ that assigns each atomic sentences a truth-value, i.e., true (1) or false (0). This can be extended to all sentences: $\nu(\varphi \wedge \psi) = 1$ iff $\nu(\varphi) = 1$ and $\nu(\psi) = 1$; $\nu(\neg\varphi) = 1$ iff $\nu(\varphi) = 0$; $\nu(\bot) = 0$; etc. Then $\Gamma \vDash \varphi$ is defined as: for all valuations ν, if $\nu(\psi) = 1$ for all $\psi \in \Gamma$, then $\nu(\varphi) = 1$. Thus, logical consequence is truth-preservation.

Now, the completeness theorem for classical propositional logic states that: $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$. To be more precise, one often only calls the right-to-left implication 'completeness', and the left-to-right implication 'soundness'. However, soundness is easy to establish. (One just needs to check, roughly, that the finitely many axioms of the proof-system are indeed logical consequences, and that the finitely many rules of the system preserves logical consequences—so the proof-system will only ever produce logical consequences.) We take soundness for granted and want to show that completeness really is a duality result.

Let us start on the syntactic side. The proof-system naturally defines a notion of equivalence between sentences: we call two sentences φ and ψ equivalent, written $\varphi \equiv \psi$, iff both $\varphi \vdash \psi$ and $\psi \vdash \varphi$. An equivalence class of a sentence φ is the set of sentences that are equivalent to it: $[\varphi] := \{\psi : \varphi \equiv \psi\}$. Write L for the set of all equivalence classes. It also has logical structure: $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$; $\neg[\varphi] = [\neg\varphi]$, etc. L is also called the *Lindenbaum–Tarski algebra* of the logic.

Now, each a valuation ν determines a subset $F_\nu \subseteq L$: namely, those equivalence classes $[\varphi]$ with $\nu(\varphi) = 1$. Note again that F_ν has features (1)–(5): If $[\varphi] \in F_\nu$ and $[\varphi] \leqslant [\psi]$ (i.e., $[\varphi \wedge \psi] = [\varphi]$), then $\varphi \vdash \psi$, so, by soundness, $\varphi \vDash \psi$, so, since $\nu(\varphi) = 1$, also $\nu(\psi) = 1$, so $[\psi] \in F_\nu$. If $[\varphi], [\psi] \in F_\nu$, then $\nu(\varphi) = 1$ and $\nu(\psi) = 1$, so $\nu(\varphi \wedge \psi) = 1$, so $[\varphi \wedge \psi] \in F_\nu$. Etc. Conversely, if $F \subseteq L$ satisfies (1)–(5), then ν_F is a valuation mapping φ to 1 iff $[\varphi] \in F$. So, again, the set X of valuations is in bijective correspondence with the set $\overline{L}$ of subsets of L satisfying (1)–(5).

But how does completeness follow? For this, first note that subsets of L are *theories*, i.e., sets of sentences (modulo provable equivalence). Now, if $\Gamma \nvdash \varphi$, consider the deductive closure Γ' of Γ, i.e., the set of all sentences that can be derived from Γ, so also $\Gamma' \nvdash \varphi$. When we regard Γ' as a subset of L, this is, in formal terminology, a filter of L that does not intersect the ideal of all equivalence classes that imply $[\varphi]$. Now one only needs one formal result, namely Stone's Prime Filter Theorem (which we prove later on in the course), which says that we can extend this filter to a prime filter



F which still does not intersect this ideal. Then ν_F is a valuation that makes true all the premises in Γ but not the conclusion φ, hence $\Gamma \not\models \varphi$, as desired.

1.1.4 Further examples in physics and computer science

We sketch two further examples, one in physics and one in computer science.

Physics: states vs observations. Duality also is a central idea in physics (e.g. Strocchi 2008, p. 24). A physical system comes both with a *state space* X and an algebra $\mathcal{A}$ of *observations* and these two again are dual in the sense that

- the states are determined by the observations that they give rise to,
- the observations are determined by the states that give rise to them.

The observations have logical structure: in a classical (as opposed to quantum) system, observing $A \wedge B$ means observing A and observing B, observing $A \vee B$ means observing A or observing B, etc. Each state x of the system determines a set of observations: namely, those that can be made if the system is in that state. Conversely, we can also start with the algebra of observations (they are empirically more accessible anyway) and postulate the states of the system as theoretical entities corresponding to certain subsets of observations.

Computation: denotations of programs vs observable properties. Computer programs are written in a programming language, and so, much like for sentences written in a natural language, we can ask what their meaning is. The meaning of a program is called its *denotation*. For example, the denotation of a program could be the (partial) function that it computes. Domain theory is the mathematical theory to systematically describe these meanings. There again also is a side that is dual to the side of meanings, and this was a crucial discovery in the development of domain theory by Abramsky (1991, p. 16). This is the side of *observable properties* of the computer programs. For example, it could be the property that, on input $x = 3$, the program halts and outputs $f(x) = 5$. Again, we would hope for a bidirectional determination in the sense that the meaning of a program is complete determined by its observable properties, and that these observable properties are determined by the denotations that have them.

1.2 Towards characterizing duality

By now, we have an interesting stock of examples involving duality. Now it is a matter of finding a concise way to systematically describe all the different components that are involved in a duality. We will work toward doing this formally for a good part of the course. But let's already give it an informal try here.

We had the following components in the examples:

- On the one side, we have a set X, e.g., of objects, possible worlds, models, states, or denotations. We hinted at the fact that this is not just a set, but actually a *space*, i.e., it also carries a topology.
- On the other side, we have a set A, e.g., of properties, propositions, sentences (modulo provable equivalence), observations, or observable properties. This set also has logical—or algebraic—structure: conjunction ($\wedge$), disjunction ($\vee$), logical falsity ($\perp$), logical truth ($\top$), and possibly negation ($\neg$).
- And we have a way to go from the spatial side to the algebraic side, and we also have a way to go in to other direction. In particular, we have:
  - A canonical way to determine from subsets of A with certain nice features an element from X, i.e., a function $\epsilon : \overline{A} \to X$.
  - A canonical way to assign to each element from A a subset of X, i.e., a function $\eta : A \to \overline{X}$.

Finally, we want to translation manual to be *formulaic* in X and A: i.e., it should not depend on the idiosyncrasies of the specific X and A; rather, it should work for all X's and A's of the same kind. This is because we do not always know the exact nature of the two sides (the objects, possible worlds, etc.; resp., the properties, propositions, etc.). So we do not want the above data for specific X and A. Rather, we want it to hold for any X that is a candidate set for the spatial side, and for any A that is a candidate for the algebraic side. And hence we also want the ways of going back and forth between the two sides to respect the relations between these candidates for the spatial side and the algebraic side.

Formally, the two sides are best represented as so-called *categories*. On the spatial side, the category consists of the spatial candidates X, which are called the *objects* of the category, and their relations, which are called the *morphisms* of the category. Similarly, on the algebraic side, the category consists of the algebraic candidates A and their relations. Then we will see that all the above components of the duality is succinctly phrased as a *dual equivalence* between the spatial category and the algebraic category.

The key application of a duality is that it provides a precise back-and-forth translation between objects (or categories) of very different kinds. Thus, questions on one side translate to question on the other side where very different tools are available to solve the question.

1.3 Exercises

Exercise 1.a. Complete the left-out details in the main text. For example, why, for a possible world x the set of propositions F_x really satisfied properties (1)–(5). Similarly for valuations v.

Exercise 1.b. Right after the list of features (1)–(5), we asked in the margin if this list is lacking a principle concerning negation: If $a \in \mathcal{P}$ is a property, then there also is the property $\neg a$ of not having property a. It seems plausible to require that either a given object $x \in \mathcal{O}$ has a property or it does not. In other words, either $a \in F_x$ or $\neg a \in F_x$. Do you think this is plausible to require? What about vague properties? (Later we see that this if if we have a negation operator on our set of properties obeying the Boolean laws, than F being prime is equivalent to having the just mentioned negation property.)

Exercise 1.c (More of a research project than an exercise)**.** Consider to what extend the first example (objects vs properties) can be developed along the lines of formal concept analysis.

Exercise 1.d. Can you think of more structure on the set of possible worlds? For example, a relation of closeness (or comparative similarity) as in the semantics for counterfactuals? Note your ideas and come back to them once we later have learned about the topology that can be put on the set of possible worlds (as hinted at in the text above). Compare this topology to your ideas.

Exercise 1.e. For a logico-philosophical discussion of the principle of indiscernibly, see Ladyman et al. (2012). How does this inform the above philosophical discussion (section 1.1.1)? This paper is in the context of model theory, what does the above duality-theoretic perspective add?

Exercise 1.f. Can you think of more examples where a duality is involved? In cognitive science: what about concepts vs. mental states (computable theory of mind vs connectionism). Or, related, in AI: or human-interpretable concepts (symbolic) vs. states of neural networks (subsymbolic)? Or are these better seen as relations of supervenience rather than duality? What about the infamous Cartesian duality between the physical and the mental world?

Exercise 1.g. Go through the discussed examples of duality again and think about where they should be made philosophically and/or mathematically more precise.

2 The algebraic side: distributive lattices

This chapter introduces formally the algebraic side of duality, which, for us, will be distributive lattices. They are particular partial orders. So, in section 2.1, we first recall order theory (which is very useful in general). Then, in section 2.2, we define lattices as particular partial orders, and we give an equivalent definition which is more algebraic (i.e., in terms of operations that satisfy equations). In section 2.3, we define when lattices are distributive and when they even are Boolean algebras. And we end with section 2.4, where we already establish a duality between finite sets (resp. finite partial orders) on the one hand and finite Boolean algebras (resp. finite distributive lattices) on the other hand. This will provide a good idea of the more general case of Stone (resp. Priestley) duality. The main missing ingredient for the general case is topology, which will be the topic of the next chapter.

2.1 Order theory

The objects that order theory studies are known as partial orders. We define them in section 2.1.1. The 'structure-preserving' maps between partial orders are known as monotone maps. We define those, and variants thereof, in section 2.1.2.

We follow one of the keys lessons of category theory: that one not only should specify the class of objects that one studies but also the class of appropriate maps—which are called morphisms—between them. These two data then constitute a category, provided some basic axioms are satisfied (that morphisms can be composed and that there is the identity morphism). We will introduce basic notions from category theory later when we need them. For now we only foreshadow it with the 'objects' and 'morphisms' distinction.

Probably we add an extra chapter on it. If so, link to it here.

2.1.1 Objects: Partial orders

Partial orders occur everywhere: when you have a bunch of things where it makes sense to say that some are bigger (better, higher, etc.) than others. The things could be numbers with the usual sense of being bigger than.

But the things could also be the dishes offered at your go-to lunch place with the sense of 'better' given by your preferences. The formal definition goes as follows.

Definition 2.1. A *partial order* (or *partially ordered set*, or *poset*) is a pair $(P, \leqslant)$ where P is a (possibly empty) set and $\leqslant$ is a binary relation on P such that

A binary relation R on a set P is simply a subset of $P \times P = \{(a, b) : a, b \in P\}$. For $a, b \in P$, one writes aRb for $(a, b) \in R$.

1. *Reflexive*: For all $a \in P$, we have $a \leqslant a$.
2. *Transitive*: For all $a, b, c \in P$, if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.
3. *Anti-symmetric*: For all $a, b \in P$, if $a \leqslant b$ and $b \leqslant a$, then $a = b$.

If we do not require axiom 3, we speak of a *preorder*. We say $\leqslant$ is a (partial or pre-) order on P. If the order $\leqslant$ is clear from context, we often simply speak of the (partial or pre-) order P.

The name 'partial' is to indicate that not all elements need to be comparable: Formally, for $a, b \in P$, we say that a and b are *comparable*, if either $a \leqslant b$ or $b \leqslant a$; otherwise they are incomparable. If all elements are comparable, we say $(P, \leqslant)$ is a *linear* (or *total*).

Formally, the example of the numbers is $(\mathbb{N}, \leqslant)$ where $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$ and, for $n, m \in \mathbb{N}$, the relation $n \leqslant m$ is defined as: n is smaller or equal to m (equivalently, there is $k \in \mathbb{N}$ such that $n + k = m$). Hence this a linear order. In the example of your lunch place, if you have two dishes a and b that you find equally tasty—or, more precisely, none tastier than the other, i.e., a and b are incomparable—, then your preference order is only partial and not linear.

Check that this satisfies the axioms.

Every partial order in particular is a preorder, and in the other direction we can canonically turn a preorder $(P, \leqslant)$ into a partial order $(\overline{P}, \overline{\leqslant})$ as follows. For $a, b \in P$, define $a \equiv b$ as $a \leqslant b$ and $b \leqslant a$. This is an equivalence relation (reflexive, symmetric, and transitive). Equivalence classes are the sets $[a] := \{b \in P : a \equiv b\}$ for $a \in P$. The quotient of P under $\equiv$ is $\overline{P} := P/ \equiv := \{[a] : a \in P\}$. Define $[a] \overline{\leqslant} [b]$ by $a \leqslant b$ (note that this is independent of the representatives a and b). This renders $(\overline{P}, \overline{\leqslant})$ a partial order. It is also called the *poset reflection* of P. Exercise 2.a makes formally precise in what sense it is the canonical or best possible poset approximating the preorder P.

There is a nice visualization of partial orders. They are known as Hasse diagrams. An example is in figure 2.1. It depicts the partial order $(P, \leqslant)$

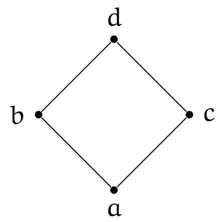


Figure 2.1: The 'diamond' as an example of a partial order.

with $P = \{a, b, c, d\}$ and

$$\leqslant := \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}.$$

This definition of the order is not particularly enlightening, but the diagram is. Its nodes are the elements of P and the edges are the minimal information to recover the order:

- if there is an edge between x and y and x is lower (on the page) than y, then $x \leqslant y$.
- we do not need to draw an edge from one node to itself because for all nodes x we have $x \leqslant x$.
- we do not need to draw edges that result from composing existing edges: for example, we have an edge from a to b and an edge from b to d, so we already know that $a \leqslant d$, hence we do not need to draw this.

More formally, the definition of a Hasse diagram of a partial order $(P, \leqslant)$ is as follows. For $a, b \in P$, we say that b *covers* a (short $a \lessdot b$) if $a \leqslant b$ and for all $c \in P$, if $a \leqslant c \leqslant b$, then $c = a$ or $c = b$. The elements of P are the nodes of the Hasse diagram, and an edge is drawn from node a to node b whenever b covers a. The direction of the edge is indicated by drawing b higher up in the diagram than a. So nodes on the same height are incomparable.

Next, some very useful concepts to talk about partial orders are the following.

They can be confusing at first, but they really are worth learning. Make sure to draw little Hasse diagrams to illustrate the concepts and how they differ from each other.

Definition 2.2. Let $(P, \leqslant)$ be a partial order and $A \subseteq P$.

- An element $b \in P$ is a *lower bound* of A if, for all $a \in A$, we have $b \leqslant a$.

- An element $b \in P$ is an *upper bound* of A if, for all $a \in A$, we have $a \leqslant b$.
- An element $c \in P$ is an *infimum* or *greatest lower bound* of A if (1) c is a lower bound of A, and (2), for all lower bounds b of A, we have $b \leqslant c$.
- An element $c \in P$ is a *supremum* or *least upper bound* of A if (1) c is an upper bound of A, and (2), for all upper bounds b of A, we have $c \leqslant b$.
- An element $b \in P$ is a *least* or *bottom* or *minimum* element of P, if, for all $a \in P$, we have $b \leqslant a$ (i.e., b is the supremum of $A = \emptyset$).
- An element $b \in P$ is a *greatest* or *top* or *maximum* element of P, if, for all $a \in P$, we have $a \leqslant b$ (i.e., b is the infimum of $A = \emptyset$).
- An element $b \in P$ is *minimal* if, for all $a \in P$, if $a \leqslant b$, then $a = b$.
- An element $b \in P$ is *maximal* if, for all $a \in P$, if $b \leqslant a$, then $b = a$.
- An element $b \in P$ is *minimal in* A if (1) $b \in A$ and (2) for all $a \in A$, if $a \leqslant b$, then $a = b$.
- An element $b \in P$ is *maximal in* A if (1) $b \in A$ and (2) for all $a \in A$, if $b \leqslant a$, then $b = a$.
- A is an *upset* if for all $a, b \in P$, if $a \in A$ and $a \leqslant b$, then $b \in A$.
- A is a *downset* if for all $a, b \in P$, if $b \in A$ and $a \leqslant b$, then $a \in A$.
- A is *directed* (aka up-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $a \leqslant c$ and $b \leqslant c$. (Equivalently, all finite subsets of A have an upper bound in A.)
- A is *filtered* (aka filtering or down-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $c \leqslant a$ and $c \leqslant b$. (Equivalently, all finite subsets of A have a lower bound in A.)

(These notions also make sense in a preorder $(P, \leqslant)$, but if P is a partial order, then infimum and supremum are unique if they exist.) The infimum is denoted $\bigwedge A$, called the *meet* of A; and the supremum is denoted $\bigvee A$, called the *join* of A. If $A = \{a_1, \dots, a_n\}$ is finite and nonempty, we write $\bigwedge A = a_1 \wedge \dots \wedge a_n$ and $\bigvee A = a_1 \vee \dots \vee a_n$. In particular, $\bigwedge\{a, b\} = a \wedge b$ and $\bigvee\{a, b\} = a \vee b$. The bottom element, if it exists, is denoted $\bot$ or 0; and the top element by $\top$ or 1. We write $\min(A)$ (resp. $\max(A)$) for

It is a good exercise to prove this.

the elements that are minimal (resp. maximal) in A. A *directed join* is the supremum of a directed set.

Partial orders where various suprema and infima exist get special names. For example, *lattices* (which we study in the next section) are partial orders where all finite subsets have an infimum and a supremum; *complete lattices* are partial orders where all subsets have an infimum and a supremum; *directed-complete partial orders* (*dcop*'s) are partial orders where all directed subsets have a supremum.

Finally, one useful operation on preorders is that we can 'turn them upside down' and get another preorder. Formally, if $(P, \leqslant)$ is a preorder, define the preorder $\leqslant'$ on P by $a \leqslant' b$ iff $b \leqslant a$. We write P^{op} for this preorder.

Verify that this again is a preorder (resp. partial order), and draw some Hasse diagram example to see that this really turns things upside down.

2.1.2 Morphisms: Monotone maps

What maps between partial orders should be considered to be 'structure preserving'? Surely they should preserve the order structure. This yields the concept of a monotone map, and is the standard choice. But there also are other ones, which we mention as well.

We consider the words 'map' and 'function' as synonymous.

Definition 2.3. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be two preorders and $f : P \to Q$ a function. We say f is

- *monotone* or *order preserving* if, for all $a, b \in P$, if $a \leqslant_P b$, then $f(a) \leqslant_Q f(b)$.
- *order reflecting* if, for all $a, b \in P$, if $f(a) \leqslant_Q f(b)$, then $a \leqslant_P b$.
- an *order-embedding* if f is both order preserving and order reflecting.
- an *order-isomorphism* if f is monotone with a monotone inverse (further comments below).

Order-embeddings between posets are injective, but the converse fails (i.e., there are injective order preserving maps between posets which are not order-embeddings.

If P and Q are posets, an equivalent condition for f being an order-isomorphism is that f is a surjective order-embedding. (It's a good exercise to verify this.) In practice, this is often easier to check, although the definition via a monotone inverse better captures the (category-theoretic) concept of an isomorphism. In full, the latter says: A monotone function $f : P \to Q$ between two preorders is an order-isomorphism if there is a monotone function $g : Q \to P$ such that

- for all $a \in P$, we have $a = g(f(a))$, i.e., a is the g-inverse of $f(a)$ (in short, $\mathsf{id}_P = g \circ f$), and

Here id_X denotes the identity function on set X. And if $f : X \to Y$ and $g : Y \to Z$ are functions, $g \circ f$ (g after f) denotes their composition, which maps $x \in X$ to $g(f(x)) \in Z$.

- for all $b \in Q$, we have $f(g(b)) = b$, i.e., mapping the g-inverse of b along f yields b (in short, $f \circ g = \mathrm{id}_Q$).

If two preorders are isomorphic (i.e., there is an order isomorphism between them), we can consider them to be essentially identical. This is difficult to achieve, so it makes sense to look for a *generalization* of the concept of an isomorphism. The key idea is to still require a monotone function $g : Q \to P$ in the other direction, but it need not be the *true* inverse but only the best possible *approximation* to an inverse:

- for all $a \in P$, we have $a \leqslant_P g(f(a))$, i.e., the g-inverse of $f(a)$ is at least as good as a, and
- for all $b \in Q$, we have $f(g(b)) \leqslant_Q b$, i.e., mapping the g-inverse of b along f approximates b.

Exercise 2.b shows why this approximation then really is best possible; and it also provides the following equivalent definition.

This is an advanced concept. Give yourself the time to let it sink in by coming back to it over and over again.

Definition 2.4. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be preorders, and let $f : P \to Q$ and $g : Q \to P$ be monotone functions. The pair (f, g) is called an *adjunction*, with f the *left* or *lower adjoint* and g the *right* or *upper adjoint*, if, for all $a \in P$ and $b \in Q$,

$$f(a) \leqslant_Q b \text{ iff } a \leqslant_P g(b).$$

Note that f occurs on the left of '$\leqslant$' and g on the right.

We also write this as $l : P \leftrightarrows Q : u$. An adjunction between P^{op} and Q is called a *Galois connection* or *contravariant adjunction*.

It is best to illustrate this abstract concept with examples. An important template of how Galois connections arise is the following (which includes the instance coining them).

Lemma 2.5. *Let $R \subseteq X \times Y$ be a relation between two sets. For any $a \subseteq X$ and $b \subseteq Y$, define*

$$u(a) := \{y \in Y : \forall x \in a. xRy\} \subseteq Y$$
$$l(b) := \{x \in X : \forall y \in b. xRy\} \subseteq X$$

Then $l : \mathcal{P}(Y) \leftrightarrows \mathcal{P}(X) : u$ forms a Galois connection between the posets $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$, i.e., for any $b \subseteq Y$ and $a \subseteq X$, we have $a \subseteq l(b)$ (i.e., $l(b) \subseteq^{\mathrm{op}} a$) iff $b \subseteq u(a)$.

Here $\mathcal{P}(X)$ is the set of all subsets of the set X.

Proof. (⇒). Assume $a \subseteq l(b)$. To show $b \subseteq u(a)$, let $y \in b$ and show $y \in u(a)$. So let $x \in a$ and show xRy. By the assumption, $x \in l(b)$, so for our $y \in b$ we have xRy.

(⇐). Assume $b \subseteq u(a)$. To show $a \subseteq l(b)$, let $x \in a$ and show $x \in l(b)$. So let $y \in b$ and show xRy. By the assumption, $y \in u(a)$, so for our $x \in a$ we have xRy. □

Here are three instances of this lemma.

1. Maybe you know the name 'Galois' from the theory of fields in algebra. Then you know Galois theory as relating fields to groups (and showing why quintic equations cannot be solved). This connection arises via the above lemma from the relation R between the set X of subfields of a given field and the set Y of automorphisms of this field, which relates a subfield to the automorphisms which are the identity on this subfield.

But you don't need to know this for the course. If you'd like an accessible introduction, have a look, e.g., at this or this video, or at these great lecture notes by Tom Leinster.

2. If X is a set and $R \subseteq X \times X$ is a preorder, then $u(a)$ is the set of upper bounds of $a \subseteq X$, and $l(b)$ is the set of lower bounds of $b \subseteq X$.

3. Consider a class of structures $\mathcal{C}$ (in, say, a first-order signature) and a class $\mathcal{F}$ of formulas (of this signature). Let $\vDash$ be the *interpretation* relation: For $M \in \mathcal{C}$ and $\varphi \in \mathcal{F}$ means that structure M makes true formula φ. Then for a set of models a, $u(a)$ is the theory of a, i.e., the set of formulas that are true in all those models. And for a a theory $b \subseteq \mathcal{F}$, $l(b)$ is the class of models of b, i.e., the set of models which make true all the sentences in b.

Also recall the examples from section 1.1.

2.2 Lattices

In this section, we define lattices as particular partial orders (and provide an equivalent algebraic definition), we define the appropriate morphisms between lattices, and we discuss some basic constructions with lattices.

2.2.1 Objects: lattices

The order-theoretic definition of a lattice goes as follows.

Definition 2.6 (Lattice, order-theoretic)**.** A *(bounded) lattice* is a partial order L in which every finite subset has a supremum and an infimum.

For example, the diamond of figure 2.1 is a lattice. Some comments:

1. In fact, it is enough that the empty set and all two-element sets have suprema and infima.

As an exercise, prove this.

2. Often a lattice is defined as a partial order in which all binary suprema and infima exist (i.e., those of two-element sets), and a bounded lattice is a lattice where also the supremum and infimum of the empty set exits (i.e., which a have a least and a greatest element). Here we assume all lattices to be bounded, because this is more convenient for duality theory. Hence we drop the word 'bounded' (unless we want to stress this assumption). A non necessarily bounded lattice can always be bounded by adding a new top and bottom element.

3. A complete lattice is a partial order in which all subsets have suprema and infima. In fact, for this it is enough that every subset has a supremum.

Prove this. (Hint: think about the supremum of all lower bounds.)

Alternatively, lattices are also defined algebraically (i.e., in terms of operations satisfying certain equations). Interestingly, these two definitions are equivalent, as we will show afterward.

Definition 2.7 (Lattice, algebraic)**.** A lattice is a tuple $(L, \vee, \wedge, \perp, \top)$ where $\vee$ (pronounced *join*) and $\wedge$ (pronounced *meet*) are binary operations on L (i.e., functions $L \times L \to L$), and $\perp$ (pronounced *bottom*) and $\top$ (pronounced *top*) are elements of L, such that the following axioms holds:

1. *commutative*: for all $a, b \in L$, we have $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.

2. *associative*: for all $a, b, c \in L$, we have $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

3. *idempotent*: for all $a \in L$, we have $a \vee a = a$ and $a \wedge a = a$.

4. *absorption*: for all $a, b \in L$, we have $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

5. *neutrality*: for all $a \in L$, we have $\perp \vee a = a$ and $\top \wedge a = a$.

For example, if X is a set, then the powerset 2^X forms a lattice in this algebraic sense with union $\cup$ as join, intersection $\cap$ as meet, $\emptyset$ as bottom, and X as top. This also provides my mnemonic for remembering what 'join' and what 'meet' is. Think of X as a set of propositions, and let $a \in 2^X$ be the beliefs (opinions, values, etc.) that *A*lice holds, and let $b \in 2^X$ be the beliefs that *B*ob holds. Then the meet of a and b—i.e., $a \wedge b = a \cap b$—is where Alice and Bob can meet: the common (meeting) ground, the set of

Though I'm happy to learn about a better one :-)

beliefs they agree on. And the join of a and b—i.e., $a \vee b = a \cup b$—is the result of joining Alice and Bob together: their joint beliefs, taking together all of their beliefs even if incoherent.

The equivalence of the two definitions is made precise in the following theorem. Exercise 2.c asks you to prove it: that is a bit tedious, but quite instructive.

Theorem 2.8. 1. *Given a lattice* $(L, \vee, \wedge, \bot, \top)$ *according to the algebraic definition, define* $a \leqslant_L b$ *as* $a \wedge b = a$*. Then* $(L, \leqslant_L)$ *is a partial order which is a lattice according to the order-theoretic definition, with binary suprema and infima being given by* $\vee$ *and* $\wedge$*.*

2. *Given a lattice* $(L, \leqslant)$ *according to the order-theoretic definition, define the binary operations* $\vee$ *and* $\wedge$ *as binary supremum and infimum, and take* $\bot$ *and* $\top$ *to be the least and greatest element of* L*. Then* $(L, \vee, \wedge, \bot, \top)$ *is a lattice according to the algebraic definition, with* $a \wedge b = a$ *iff* $a \leqslant b$ *iff* $a \vee b = b$*.*

From now on, we will often just speak of a lattice L and both use its order-theoretic definition (taking $\leqslant$ to be implicitly given) and its algebraic definitions (taking $\vee, \wedge, \bot, \top$ to be implicitly given).

Finally, in some situations we might only have one of the two binary operations: then we speak of a semilattice. Formally, a *semilattice* is a structure $(L, \cdot, 1)$, where $\cdot$ is a commutative, associative, and idempotent binary operation on L, and 1 is a neutral element for the operation. The operation $\cdot$ can then either be seen as the binary infimum for the partial order defined by $a \leqslant b$ iff $a \cdot b = a$ (the join semilattice), or as the binary supremum for the opposite partial order defined by $a \leqslant b$ iff $a \cdot b = b$ (the meet semilattice).

2.2.2 Morphisms: lattice homomorphisms

The appropriate structure preserving map between lattices is the following:

Definition 2.9. A function $f : L \to M$ between lattices is a lattice homomorphism if it preserves all the lattice operations, i.e.,

1. for all $a, b \in L$, we have $f(a \vee_L b) = f(a) \vee_M f(b)$
2. for all $a, b \in L$, we have $f(a \wedge_L b) = f(a) \wedge_M f(b)$
3. $f(\bot_L) = \bot_M$
4. $f(\top_L) = \top_M$

Note that lattice homomorphisms are always order preserving, and injective lattice homomorphisms are order-embeddings. An injective lattice homomorphism is called a *lattice embedding*. Bijective lattice homomorphisms are order-isomorphisms and are called *lattice isomorphisms*.

Prove this.

If a function $f : L \to M$ between lattices preserves $\bot$ and $\vee$, then it preserves all finite joins. This does, in general, *not* imply any preservation of arbitrary existing joins or preservation of infima. The analog statement is true for $\top$ and $\wedge$ and preservation of all finite meets.

Prove this.

2.2.3 Constructions: products, sublattices, homomorphic images, congruences

We introduce several common constructions on lattices. They are common algebraic operations that you might have seen already in other contexts (e.g., group theory); and, in any case, they are worth knowing as they come up quite often.

Products. Given a family $(L_i)_{i \in I}$ of lattices, we can define a lattice $L = \prod_{i \in I} L_i$ on the Cartesian product where the operations are defined component-wise: e.g., for $a = (a_i)_i$ and $b = (b_i)_i$ in L, we define $a \leqslant_L b$ as $\forall i \in I : a_i \leqslant_{L_i} b_i$, and $(a \wedge b)_i = a_i \wedge b_i$ (similarly for $\vee$), and $(\bot_L)_i = \bot_{L_i}$ (similarly for $\top$). The projection maps $\pi_i : L \to L_i$, which map $a = (a_i)_i$ to a_i, is a surjective lattice homomorphism.

Recall that the Cartesian product of a family of sets is the set of functions a that map each $i \in I$ to an element $f(i) \in L_i$. We often write such a function as $a = (a_i)_{i \in I}$.

Sublattices. A sublattice of a lattice L is a subset L' of L that contains $\bot$ and $\top$ and that is closed under $\wedge$ and $\vee$ (i.e., if $a, b \in L'$, then $a \wedge b, a \vee b \in L'$). Then L' is a bounded lattice in its own right and the inclusion map $\iota : L' \to L$, which maps $a \in L'$ to $a \in L$, is a lattice embedding. If we do not require $\bot$ and $\top$ to be in L', we speak of an *unbounded sublattice*. And if we require L' to be closed under all suprema and infima, we call it a *complete sublattice*. If $f : L \to M$ is a lattice homomorphism, then the direct image $L' := f[L] = \{f(a) : a \in L\}$ is a sublattice of the lattice M.

Homomorphic images. A lattice L' is a *homomorphic image* of a lattice L if there is a surjective lattice homomorphism $f : L \to L'$.

Birkhoff's famous theorem in universal algebra says that a class of algebraic structures (like lattices) is closed under Homomorphic images, Subalgebras, and Products iff it is definable by equations (hence aka 'HSP theorem').

Congruences. A congruence on a lattice L is an equivalence relation ϑ on L that respects the lattice operations, i.e., for all $a, a', b, b' \in L$, if $a\vartheta a'$ and $b\vartheta b'$, then also $a \vee b \vartheta a' \vee b'$ and $a \wedge b \vartheta a' \wedge b'$. For an intuitive example, think of the elements of L as propositions and of ϑ as having the same subject matter. The quotient L/ϑ carries a unique lattice structure that turns the quotient map $p : L \to L/\vartheta$, which maps $a \in L$ to its equivalence class $[a]_\vartheta$ under ϑ, into a lattice homomorphism; concretely, this lattice structure is given by $[a]_\vartheta \vee [b]_\vartheta := [a \vee b]_\vartheta$ (similarly for $\wedge$) with bottom element $[\bot]_\vartheta$ (similarly for $\top$). Note how this is reminiscent of the Lindenbaum–Tarski algebra from the introduction (section 1.1.3).

The first isomorphism theorem for lattices. This says that any lattice homomorphism $f : L \to M$ can be factored as a surjective lattice homomorphism p followed by a lattice embedding e (i.e., $f = e \circ p$). These are given as follows. The *kernel* of f is the congruence relation

$$\ker f := \{(a, a') \in L \times L : f(a) = f(a')\}.$$

Choose $p : L \to L/\ker f$ (mapping a to $[a]$) and $e : L/\ker f \to M$ (mapping $[a]$ to $f(a)$). In particular, $L/\ker f$ is isomorphic to $f[L]$ (take $M := f[L]$, so e also is surjective); hence the homomorphic images of L are, up to isomorphism, the quotients of L.

To exciting thing about this is that lattice homomorphism can be very complicated, but this tells us that they can be broken down into two much simpler things: surjective lattice homomorphisms and injective lattice homomorphisms!

Again, the quotients of L intuitively are much simpler: to determine them, we only have to look at L, while for homomorphic images we also need to consider other lattices M.

2.3 Distributive lattices and Boolean algebras

We get further subclasses of lattices by requiring that $\vee$ and $\wedge$ interact nicely, which is made precise as distributive lattices (section 2.3.1), and by additionally requiring that there is a sense of negation, which is made precise as Boolean algebras (section 2.3.2).

2.3.1 Distributive lattices

The idea $\vee$ and $\wedge$ interact nicely is made precise as follows.

Definition 2.10. A lattice L is distributive if,

$$\forall a, b, c \in L : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \tag{2.1}$$

or, equivalently,

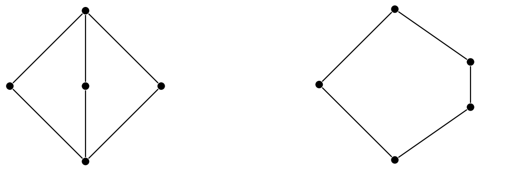
$$\forall a, b, c \in L : a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \tag{2.2}$$

Cf. distributivity from high school: $x \times (y + z) = (x \times y) + (x \times z)$

Proving the equivalence of 2.1 and 2.2 is a good exercise!

For example, the four diamond from figure 2.1 is distributive (check why), as is any powerset 2^X.

The equivalence of 2.1 and 2.2 implies that L is distributive iff L^{op} is distributive. So distributivity is a so-called self-dual property. Moreover, homomorphic images, sublattices, and products of distributive lattices are again distributive. (This follows from the 'HSP theorem' and the fact that distributive lattices are defined equationally.)



The diamond M_3 The pentagon N_5

Figure 2.2: The forbidden substructures for distributivity.

Again important special cases are as follows: A *frame* is a complete lattice L satisfying the join infinite distributive law (JID)

In case you have heard of this: A frame is the same thing as a complete Heyting algebra, but their respective choice of morphisms differ.

$$\text{for any } a \in L \text{ and } B \subseteq L,\ a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b). \tag{2.3}$$

In a distributive lattice this, in general, only holds for all *finite* $B \subseteq L$.

A seemingly magic characterization of distributive lattices is the following.

Theorem 2.11 (The M_3–N_5 theorem)**.** *Let L be a lattice. Then L is distributive iff L does not contain an unbounded sublattice which is isomorphic to M_3 or N_5, depicted in figure 2.2.*

For a proof, see, e.g., Davey and Pristley (2002, 89 ff.).

2.3.2 Boolean algebras

So far, we have seen the order $\leqslant$ and the operations $\vee$ and $\wedge$ in a lattice, which act like implication, disjunction, and conjunction, respectively. So you might have wondered: what about negation? Especially since this also played a role in our motivating introduction (chapter 1). The (or, more precisely, a) idea of negation is made precise as follows.

Definition 2.12. Let L be a lattice and a an element of L. A *complement* of a is an element b of L such that $a \wedge b = \bot$ and $a \vee b = \top$. A *Boolean algebra* is a distributive lattice in which every element has a complement. The complement of an element a in a distributive lattice is unique, if it is exist, an denoted $\neg a$.

For example, again the four diamond from figure 2.1 is a Boolean algebra (check why), as is any powerset 2^X. Some further comments:

1. Usually, the negation is then taken into the signature: so a Boolean algebra is a tuple $(B, \wedge, \vee, \bot, \top, \neg)$ such that $(B, \wedge, \vee, \bot, \top)$ is a distributive lattice and $\neg : B \to B$ a unary function such that, for all $a \in B$, we have $a \wedge \neg a = \bot$ and $a \vee \neg a = \top$.

2. But if we have an additional operation around, shouldn't we require the morphisms to preserve it? Fortunately, they already do: If $f : B \to A$ is a lattice homomorphism between Boolean algebras, then, for all $a \in B$, we have $f(\neg a) = \neg f(a)$. We often still refer to them as *Boolean algebra homomorphisms* just to emphasize that we are dealing with Boolean algebras.

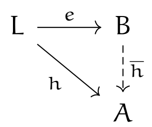
The fact that we can use the same morphisms is expressed in categorical terms as the category of Boolean algebras and Boolean algebra homomorphisms being a full *(as opposed to any) subcategory of the category of distributive lattices and lattice homomorphisms.*

3. However, with the notion of a sublattice we need to be more careful: A Boolean algebra may have many sublattices that themselves are not Boolean algebras; so by a (*Boolean*) *subalgebra* of a Boolean algebra B we mean a sublattice which is also closed under $\neg$.

4. If you like ring theory, a Boolean algebra can equivalently be defined as a commutative ring with unit in which all elements are idempotent, see exercise 2.d.

5. There is a best way to turn a distributive lattice L into a Boolean algebra B. This B is called the *Boolean envelope* or *free Boolean extension* of L. More precisely, this means that for every distributive lattice L there is a Boolean algebra B and an injective homomorphism $e : L \to B$ such that for any other lattice homomorphism $h : L \to A$ into a Boolean algebra A, there is a unique Boolean algebra homomorphism $\overline{h} : B \to A$ such that $\overline{h} \circ e = h$. As a diagram:

In categorical terms this means the category of Boolean algebras is a full reflective *subcategory of the category of distributive lattices.*



We later will prove this theorem using duality theory.

2.4 Duality for finite distributive lattices and finite Boolean algebras

In this section, we prove a 'baby version' of the duality result that we are working toward. Of course, the baby version will follow from the full version, but we prove it now already mostly for pedagogical reasons (1) to already reap some benefits of the build-up of theory so far and (2) to already get used to how a duality theorem looks like.

So let's see to what extent we can formalize the intuitions from the introduction (chapter 1). For concreteness, let's work with the first example of ‘properties vs objects’ (section 1.1.1), but all we will say will also work for the other examples; so feel free to swap things to your favorite example.

2.4.1 From properties/lattice to recovered objects/spaces and back

So let L be a *finite* set of properties with the logical operations $\vee$ (disjunction of properties), $\wedge$ (conjunction of properties), $\perp$ (the inconsistent property), $\top$ (the trivial property). So, plausibly, L is a finite distributive lattice. Recall how we recovered what the set X of possible objects must be: They are subsets F of L with the following properties, for all $a, b \in L$:

1. If $a \in F$ and $a \leqslant b$, then $b \in F$.
2. If $a, b \in F$, then $a \wedge b \in F$.
3. If $a \vee b \in F$, then either $a \in F$ or $b \in F$.
4. $\top \in F$
5. $\perp \notin F$.

Call a subset $F \subseteq L$ with properties 1, 2, and 4 a *filter*. A filter is *proper* if it also has property 5. And it is *prime* if it additionally has property 3.

Since L is finite, filters can be described more concretely: they all must have a least element $c \in F$, so they are of the form $\uparrow c = \{a \in L : c \leqslant a\}$. Here is why: Since F is nonempty, let $c_0 \in F$. If c_0 is not the least element of F, there must be $c_1' \in F$ with $c_0 \nleqslant c_1'$. So $c_1 := c_0 \wedge c_1' \in F$ which is strictly below c_0 (as infimum it is below, and if we had $c_0 = c_1 = c_0 \wedge c_1'$, then $c_0 \leqslant c_1'$). If c_1 is not the least element of F, there again must be $c_2' \in F$ with $c_1 \nleqslant c_2'$, so $c_1 > c_2 := c_1 \wedge c_2' \in F$. And so on. However, since $F \subseteq L$ is finite, this process cannot go on forever, and whenever it stops, we have found the least element of F.

So a filter F can more concisely be described by its least element c. What, then, are our desired prime filters? To say that F is proper is to say that $c \neq \perp$, and to say that F is prime is to say that also the least element c has the following property:

$$\forall a, b \in L : c \leqslant a \vee b \Rightarrow c \leqslant a \text{ or } c \leqslant b.$$

An element c of L that is not $\perp$ and that has this property is called *join-prime*.

So, our recovered version $\mathcal{J}(L)$ of the set X of possible objects is the set of join-prime elements of L. So our recovered objects $\mathcal{J}(L) \subseteq L$ are ordered by

NB: The book defines $\mathcal{J}(L)$ to be the join-irreducible *elements of L. For general lattices, these differ from join-prime elements; but for distributive lattices, they coincide (and this is equivalent to being distributive). See exercise 2.f.*

the order inherited from L. And this makes sense: also our real objects X are ordered by *generalization*: For $x, y \in X$, say y is more or equally general compared to x (written $y \succeq x$), if for all properties $a \in L$, if y has a, then also x has a (but x might have more properties, and thus be more special). The properties of a recovered object $c \in \mathcal{J}(L)$ are precisely the elements of the filter it represents, i.e., the $a \in \uparrow c$. So for two recovered objects $c, d \in \mathcal{J}(L)$, we have that d is more general than c (i.e., $d \succeq c$) precisely if for all $a \in L$, if $a \geqslant d$, then $a \geqslant c$. But this is equivalent to $d \geqslant c$. So the generalization order $\succeq$ of $\mathcal{J}(L)$ really is the order $\geqslant$ inherited from L.

The opposite of the generalization order is the specialization order (which might be better known and which we see in chapter 3): y is more special than x iff x is more general than y iff any property of x is also a property of y.

Okay, so far we started with a finite distributive lattice L of properties and recovered the finite set of possible objects as $\mathcal{J}(L)$. But what if we go in the other direction and start with a finite set of possible objects X, which—as we just saw—comes with a generalization order $\preceq$, and we want to recover the properties L?

So let X be a finite set of possible objects with generalization order $\preceq$. What are the properties of X? They are described by their extension, i.e., the subset a of X consisting of the objects in X which have this property. These properties will respect the generalization order: For $x, y \in X$, if y is more general than x, i.e., $x \preceq y$, then if y has property a (i.e., $y \in a$), then also x has property a (i.e., $x \in a$). So a is a downset of the generalization order! So we identify the recovered properties of X with the downsets of $(X, \preceq)$. We write $\mathcal{D}(X)$ for the set of downsets of X. It's not hard to check that $\mathcal{D}(X)$ is a (even complete) sublattice of the powerset lattice 2^X, and hence also distributive.

See exercise 2.g.

2.4.2 The recovered lattices and spaces are isomorphic to the original ones

Next we said that we would expect (1) that the set of objects with a generalization order is isomorphic to the objects recovered from the recovered properties (the 'double dual' properties), and that (2) the set of properties is isomorphic to the properties recovered from the recovered objects (the 'double dual' objects). Let's verify these in turn.

Concerning the isomorphism between double dual and original objects, recall from section 1.1.1, that our guiding principle was this: Each set of properties $F \subseteq L$ that is a prime filter determines an object $x \in X$, namely, the unique object having exactly the properties in F. Translated into our formal setting, the recovered objects A (corresponding to F) are join-prime elements of the recovered properties, which in turn are downsets of X (the set of downsets corresponds to L). The set of properties encoded by A is the set of all downsets D that contain A. So an original object $x \in X$ has all those properties if it is in all D, i.e., it is in A. To have a canonical choice, we would want to pick the most general object of A, i.e., we hope that A has a greatest element—fortunately this is the case.

Proposition 2.13. *Let $(X, \preceq)$ be a finite partial order. Then the following is well-defined and an order-isomorphism.*

$$\begin{aligned} \widehat{\cdot} : \mathcal{J}(\mathcal{D}(X)) &\to X \\ A &\mapsto \text{the greatest element of } A \end{aligned}$$

Proof. We first show that A indeed has a greatest element. Since $A \subseteq X$ is a downset, we have (writing $\downarrow a := \{x \in X : x \preceq a\}$)

$$A = \bigcup_{a \in A} \downarrow a.$$

Since X is finite, this is a finite join in the lattice $\mathcal{D}(X)$ that is (greater-or) equal to A. Since A is join-prime, there hence is $a \in A$ such that $\downarrow a \supseteq A$, so, since we have the other inclusion since $a \in A$, we have $\downarrow a = A$, hence a is the greatest element of A.

So $\widehat{\cdot}$ is a well-defined function. It remains to show that it is an order-isomorphism.

Order-preservation and -reflection: Let $A, B \in \mathcal{J}(\mathcal{D}(X))$. If $A \subseteq B$, then the greatest element a of A is in B, and hence it is below the greatest element b of B. Conversely, if $a \leqslant b$, then $A = \downarrow a \subseteq \downarrow b = B$.

Surjectivity: If $x \in X$, consider $A := \downarrow x$. This downset is join-prime: If A_1 and A_2 are two downsets of X with $A_1 \cup A_2 \supseteq A$, then x is in A_i for i either 1 or 2, and since A_i is a downset, $A = \downarrow x \subseteq A_i$. Finally, we clearly have $\widehat{A} = x$, as needed. □

Concerning the isomorphism between double dual and original properties, the guiding principle was the following. (We did not discuss it in section 1.1.1, but as the truthset function in section 1.1.2.) Each property a is mapped to its *extension*, i.e., the set of objects that have this property. Translated into our formal setting, the recovered objects are prime filters, represented as join-prime elements j, and them having property a means that a is in the filter, i.e., $j \leqslant a$.

Proposition 2.14. *Let L be a finite distributive lattice. Then the following defines*

This is known as Birkhoff's representation theorem (from 1937).

a lattice isomorphism.

$$\widehat{\cdot} : L \to \mathcal{D}(\mathcal{J}(L))$$
$$a \mapsto \widehat{a} := \{j \in \mathcal{J}(L) : j \leqslant_L a\}$$

Proof. Note that, by construction, $\widehat{a}$ is a downset of $\mathcal{J}(L)$, so this is well-defined. We show that $\widehat{\cdot}$ is a bijective lattice homomorphism.

Concerning meet, we have to show $\widehat{a \wedge b} = \widehat{a} \cap \widehat{b}$. Indeed, for $j \in \mathcal{J}(L)$, we have $j \leqslant a \wedge b$ iff $j \leqslant a$ and $j \leqslant b$.

Concerning join, we have to show $\widehat{a \vee b} = \widehat{a} \cap \widehat{b}$. Indeed, for $j \in \mathcal{J}(L)$, we have that if $j \leqslant a$ or $j \leqslant b$, also $j \leqslant a \vee b$. And if $j \leqslant a$ and $j \leqslant b$, then, since j is join-prime, either $j \leqslant a$ or $j \leqslant b$.

Concerning bottom, we have to show $\widehat{\bot} = \emptyset$. Indeed, otherwise there is $j \in \mathcal{J}(L)$ with $j \leqslant \bot$, so $j = \bot$, but the bottom element never can be join-prime.

Concerning top, we have to show $\widehat{\top} = \mathcal{J}(L)$. Indeed, for any $j \in \mathcal{J}(L)$, we have $j \leqslant \top$.

Concerning injectivity, if $a \neq b$, then either $a \nleqslant b$ or $b \nleqslant a$. Without loss of generality, assume $a \nleqslant b$. We find $j \in \mathcal{J}(L)$ such that $j \leqslant a$ but $j \nleqslant b$, hence $\widehat{a} \neq \widehat{b}$. Indeed, consider the set $A := \{c \in L : c \leqslant a \text{ and } c \nleqslant b\}$. It is nonempty, since $a \in A$. Since L is finite, it has a minimal element j. So it remains to show that j is join-prime. Now, j cannot be $\bot$, since $\bot \leqslant b$. So assume $j \leqslant c \vee c'$, i.e., $j = j \wedge (c \vee c') = (j \wedge c) \vee (j \wedge c')$. We have that either $j \wedge c \nleqslant b$ or $j \wedge c' \nleqslant b$, because otherwise b is an upper bound of both, so $j = (j \wedge c) \vee (j \wedge c') \leqslant b$. Without loss of generality, say $j \wedge c \nleqslant b$. Since also $j \wedge c \leqslant j \leqslant a$, we have $j \wedge c \in A$. Since $j \wedge c \leqslant j$ and j is minimal in A, we have $j \wedge c = j$, so $j \leqslant c$, as needed.

Concerning surjectivity, let $A \in \mathcal{D}(\mathcal{J}(L))$ and find $a \in L$ such that $\widehat{a} = A$. Since A is finite, $a := \bigvee A \in L$ exists. To show $\widehat{a} = A$, let $j \in \mathcal{J}(L)$. Indeed, we have: $j \in \widehat{a}$ iff $j \leqslant \bigvee A$ iff (since j is join-prime) $j \leqslant b$ for some $b \in A$ iff (since A is a downset) $j \in A$. □

2.4.3 Also including morphisms

Finally, to complete the duality, we also want a correspondence between the morphisms of the algebraic and the spatial side. After all, we should not only be able to relate objects, but also their interconnections!

More concretely, we now know that there is an exact correspondence between finite partial orders X (the spatial side) and finite distributive lattices L (the algebraic side): we can relate X to the lattice $\mathcal{D}(X)$, and this is 'bijective' correspondence because every lattice L is of this form (since $L \cong \mathcal{D}(\mathcal{J}(L))$) and distinct posets/spaces are related to distinct lattices (if $\mathcal{D}(X) \cong \mathcal{D}(Y)$, then $X \cong \mathcal{J}(\mathcal{D}(X)) \cong \mathcal{J}(\mathcal{D}(Y)) \cong Y$).

Technically, this relates class-many structures to class-many structures, so we cannot speak of a bijective functions, and it is only up to isomorphism—hence the quotation marks.

So, do we have a similar bijective correspodence for the morphisms? Do order-preserving maps between posets correspond bijectively to lattice homomorphisms between their dual lattices? Yes, as the following shows.

Proposition 2.15. *Let* $(X, \preceq)$ *and* $(Y, \preceq)$ *be finite partial orders. Let* $f : X \to Y$ *be order-preserving. Then*

$$\begin{aligned} \mathcal{D}(f) : \mathcal{D}(Y) &\to \mathcal{D}(X) \\ D &\mapsto f^{-1}(D) \end{aligned}$$

is a lattice homomorphism. And if $h : \mathcal{D}(Y) \to \mathcal{D}(X)$ *is a lattice homomorphism, there is a unique order-preserving function* $f : X \to Y$ *such that* $\mathcal{D}(f) = h$*.*

Before we consider the proof, some comments. Note that f and $\mathcal{D}(f)$ go in opposite directions: f goes from X to Y, but $\mathcal{D}(f)$ goes from $\mathcal{D}(Y)$ to $\mathcal{D}(X)$. This is a characteristic feature of dualities: it is precisely the reason why, in the category-theoretical terminology, we say that we have a *dual* equivalence and not just an equivalence between the spatial and the algebraic side. One reason is that the inverse image map $f^{-1}(\cdot)$ has much better preservation properties than the direct image map $f[\cdot]$, which would go in the same direction: For example, the inverse of an intersection is the intersection of the inverses, but the image of an intersection need not be the intersection of the images.

We won't consider the proof in full detail, but we stress the use of an adjunction in it.

Proof sketch. Proving that $\mathcal{D}(f)$ is a lattice homomorphism is a very good exercise it precisely brings about the strong preservation properties of the inverse image map. For the second claim, one convenient fact about homomorphisms h between finite lattices is this: Since they preserve all finite meets, and since there are only finitely many meets, they in fact preserve all meets. We saw in exercise 2.b that this is characteristic of upper adjoints, and one can show that h indeed must have a lower adjoint g. Since h also preserves joins, it is not difficult to show—using the defining feature of an adjunction—that then g maps join-prime elements to join-prime elements.

See exercise 2.h.

With this, we can define $f : X \to Y$ as follows. Given $x \in X$, we find $y = f(x)$ as follows: The downset $\downarrow x \in \mathcal{D}(X)$ is join-prime, so $g(\downarrow x)$ is join-prime in $\mathcal{D}(Y)$, so it is of the form $\downarrow y$, with $y \in Y$, and we choose this y as the value of x under f.

That f is order preserving can be seen from g being order preserving. Finally, for $E \in \mathcal{D}(Y)$, we need to show $h(E) = f^{-1}(E)$. Indeed, for $x \in X$, we have, by the adjunction, that

$$x \in h(E) \overset{\text{downset}}{\Longleftrightarrow} {\downarrow}x \subseteq h(E) \overset{\text{adjunction}}{\Longleftrightarrow} g({\downarrow}x) \subseteq E$$
$$\overset{\text{def. of } f}{\Longleftrightarrow} {\downarrow}f(x) \subseteq E \overset{\text{downset}}{\Longleftrightarrow} f(x) \subseteq E.$$

For uniqueness, if $f, f' : X \to Y$ are such that $\mathcal{D}(f) = \mathcal{D}(f')$, we leave it as an exercise to show that $f = f'$. □

See exercise 2.i.

2.4.4 The duality and the special case of Boolean algebras

If we take everything together, we get our desired duality. With $\mathcal{D}$, we have a way to go from the spatial side to the algebraic side: map a poset X to the lattice $\mathcal{D}(X)$, and a order preserving map $f : X \to Y$ to lattice homomorphism $\mathcal{D}(f) : \mathcal{D}(Y) \to \mathcal{D}(X)$. With $\mathcal{J}$, we have a way to go from the algebraic side to the spatial side: map a lattice L to the poset $\mathcal{J}(L)$, and a lattice homomorphism $h : L \to M$ to the unique order preserving map $f : \mathcal{J}(L) \to \mathcal{J}(M)$ with $\mathcal{D}(f) = h$ (up to isomorphism of the domain and codomain). And going back and forth in this way cancels out: $\mathcal{D}(\mathcal{J}(L)) \cong L$ and $\mathcal{J}(\mathcal{D}(X)) \cong X$. To summarize:

Theorem 2.16. *The facts expressed by propositions 2.13–2.15 is expressed category-theoretically as: The functors $\mathcal{D}$ and $\mathcal{J}$ constitute a duality between the category DL_f of finite distributive lattices with lattice homomorphisms and the category Pos_f of finite posets with order-preserving functions.*

It remains to see what happens if we restrict us from distributive lattices to Boolean algebras: i.e., distributive lattice where we also have a notion of complement. What are the corresponding spaces?

Let's see: If L is a finite Boolean algebra, what can we say about the dual space $(X, \preceq) := \mathcal{J}(L)$? Specifically, how does the partial order look like? Surprisingly, it must be the identity relation: If a and b are join-prime elements of the Boolean algebra A with $a \leqslant b$, then $a = b$. Proof: We have $b \leqslant \top = a \vee \neg a$, so either $b \leqslant a$ or $b \leqslant \neg a$. The latter cannot be: Otherwise $a \leqslant b \leqslant \neg a$, so $a = a \wedge \neg a = \bot$, but $a \neq \bot$ qua join-prime element. Hence $b \leqslant a$. Since also $a \leqslant b$, the claim $a = b$ follows.

In fact, finite Boolean algebras are characterized by this, see Gehrke and van Gool 2023, prop. 1.26.

In short: Once the properties of our space that we consider have complements (i.e., can be negated), the generalization order on the points of the space becomes trivial.

Conversely, if $(X, \preceq)$ is a finite partial order where $\preceq$ is the identity relation, then any subset is a downset, so $\mathcal{D}(X) = 2^X$ is a Boolean algebra! So the dual spaces of finite Boolean algebras are precisely the posets where the order is the identity relation. But then we might as well just forget the order, as it adds no information. Hence our duality cuts down to a duality between finite Boolean algebras and finite sets:

Theorem 2.17. *The functors $\mathcal{D}$ and $\mathcal{J}$ cut down to a duality between the category* BA_f *of finite Boolean algebras with homomorphisms and the category* Set_f *of finite sets with functions.*

At the end of section 1.1.2, we already hinted at the fact that, to generalize this result beyond the finite, we to add some further structure on the spaces: namely, a topology. This is what we will do in the next chapter.

2.5 Exercises

Exercise 2.a. Recall that for a preorder $(P, \leqslant)$, we have defined the poset reflection $(\overline{P}, \overline{\leqslant})$. This exercise makes precise in which sense this is the best possible poset approximating the preorder $(P, \leqslant)$.

> *Exercise 1.1.5 in Gehrke and van Gool (2023), with small changes.*

1. Prove that $\equiv$ is an equivalence relation.

2. Prove that the definition of $\overline{\leqslant}$ is independent of the representatives: If $a' \in [a]$ and $b' \in [b]$, then $a \leqslant b$ iff $a' \leqslant b'$.

3. Prove that $(\overline{P}, \overline{\leqslant})$ is indeed a partial order.

4. Prove that $\overline{\leqslant}$ is the smallest partial order on $\overline{P} = P/\equiv$ such that the quotient map $f : P \to P/\equiv$, which maps a to $[a]$, is order preserving: That is, if $\leqslant'$ is another such partial order on $P/\equiv$, then $\overline{\leqslant} \subseteq \leqslant'$.

5. Prove that, for any order preserving $g : P \to Q$ into a poset Q, there exists a unique order preserving $\overline{g} : P/\equiv \to Q$ such that $\overline{g} \circ f = g$. As a diagram:

$$\begin{array}{ccc} P & \xrightarrow{\;f\;} & P/\equiv \\ & {}_{g}\searrow & \downarrow{\scriptstyle \overline{g}} \\ & & Q \end{array}$$

> *The category-theoretic formulation of this fact is: the inclusion of the category of partial orders and monotone maps in the category of preorders and monotone maps has a left adjoint. Adjoint functors can be interpreted as formalizing the idea of finding a best possible approximation.*

Think about how the last item formalizes the idea that $(\overline{P}, \overline{\leqslant})$ is the best possible poset approximating the preorder $(P, \leqslant)$.

Exercise 2.b. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be two preorders, and let $f : P \to Q$ and $g : Q \to P$ be monotone maps.

Exercise 1.1.8 in Gehrke and van Gool (2023).

1. Prove that (f, g) is an adjunction iff for all $a \in P$ we have $a \leqslant_P g(f(a))$ and for all $b \in Q$ we have $f(g(b)) \leqslant b$.

For the rest of this exercise, assume that (f, g) is an adjunction.

2. Prove that $f \circ g \circ f(a) \equiv f(a)$ and $g \circ f \circ g(b) \equiv g(b)$ for every $a \in P$ and $b \in Q$ (and $a \equiv b$ iff $a \leqslant b$ and $b \leqslant a$).
3. Conclude that, in particular, if P and Q are posets, then $fgf = f$ and $gfg = g$.
4. Prove that, if P is a poset, then for any $a \in P$, $gf(a)$ is the least element above a that lies in the image of g.
5. Formulate and prove a similar statement to the previous item about $fg(b)$, for $b \in Q$.
6. Prove that, for any subset $A \subseteq P$, if the supremum of A exists, then $f(\bigvee A) = \bigvee f(A)$ (where $f(A) = \{f(a) : a \in A\}$ is the image of A under f).
7. Prove that, for any subset $B \subseteq Q$, if the infimum of B exists, then $g(\bigwedge B) = \bigwedge g(B)$.

In words, the last two items say that *lower adjoints preserve existing suprema* and *upper adjoints preserve existing infima*.

We will see that the converse holds for complete lattices. This is a special case of the Adjoint Functor Theorem.

Exercise 2.c. Prove theorem 2.8.

Exercise 2.d. This exercise shows that Boolean algebras and Boolean rings are equivalent.

From Gehrke and van Gool 2023, ex. 1.2.13.

1. Let $(B, +, \cdot, 0, 1)$ be a Boolean ring, i.e., a commutative ring with unit in which $a \cdot a = a$ for all $a \in B$. Define $a \leqslant b$ if $a \cdot b = a$. (We often write ab for $a \cdot b$.) Prove that $\leqslant$ is a distributive lattice order on B where
   - 1 is the greatest element and 0 is the least element,
   - meet is given by ab and join is given by $a + b + ab$, and
   - every element a of has the complement $1 + a$ with respect to $\leqslant$.

   Hint: First show that $a + a = 0$ for all $a \in B$.

2. Conversely, let $(B, \wedge, \vee, \bot, \top, \neg)$ be a Boolean algebra. Define, for any $a, b \in B$,

$$a + b := (a \wedge \neg b) \vee (\neg a \wedge b) \qquad a \cdot b := a \wedge b$$
$$0 := \bot \qquad 1 := \top.$$

The operation $+$ is known as symmetric difference.

Prove that $(B, +, \cdot, 0, 1)$ is a Boolean ring.

3. Finally, show that the composition of these two assignments in either order yields the identity.

Exercise 2.e. To provide examples and non-examples about the notion of complement:

1. Show that the n-chain $\mathbf{n} = \{0, 1, \dots, n-1\}$ is a distributive lattice.

2. Show that, for $n = 2$, the two-element lattice $\mathbf{2} = \{0, 1\}$ is a Boolean algebra.

3. But for $n \geqslant 3$, $\mathbf{n}$ is not a Boolean algebra. (Complements don't always exist.)

4. Show that the distributive lattices M_3 and N_5 from figure 2.2 are not Boolean algebras. (Complements exists but are not unique.)

5. Show that the complement of an element of a distributive lattice is unique if it exists.

Exercise 2.f. This exercise explores the difference and similarities of join-prime and join-irreducible elements. Let L be a lattice. Recall that an element $a \in L$ is

- join-prime if $a \neq \bot$ and for all $b, c \in L$, if $a \leqslant b \vee c$, then either $a \leqslant b$ or $a \leqslant c$.

- join-irreducible if $a \neq \bot$ and for all $b, c \in L$, if $a = b \vee c$, then either $a = b$ or $a = c$.

Now, establish the following relationships between these concepts.

1. Show that a is join-prime iff for all finite $S \subseteq L$, if $a \leqslant \bigvee S$, then there is $s \in S$ such that $a \leqslant s$.

2. Convince yourself that you can analogously show that a is join-irreducible iff for all finite $S \subseteq L$, if $a = \bigvee S$, then there is $s \in S$ such that $a = s$.

3. Show that if a is join-prime, then a is join-irreducible.

4. Show that the converse holds if L is distributive.

5. Provide an example of a lattice L which has a join-irreducible element a which is not join-prime.

Exercise 2.g. Let X be a poset and $\mathcal{D}(X)$ the set of its downsets. Show that $\mathcal{D}(X)$ is a sublattice of 2^X. In particular, intersections and unions of downsets are again downsets.

Exercise 2.h. Let $f : X \to Y$ be a function between two sets X and Y. Show that the function from 2^Y to 2^X defined by

$$B \mapsto f^{-1}(B)$$

is a Boolean algebra homomorphism. Show that if f additionally is a monotone function between two posets X and Y, then this mapping restricts to a lattice homomorphism $\mathcal{D}(Y) \to \mathcal{D}(X)$.

Exercise 2.i. In the context of proposition 2.15, show that if $f, f' : X \to Y$ are such that $\mathcal{D}(f) = \mathcal{D}(f')$, then $f = f'$.

3 The spatial side: topological spaces

This chapter introduces formally the spatial side of duality, which, for us, will be certain ordered topological spaces, i.e., topological spaces that also have a partial order on their points. This naturally structures this chapter: In section 3.1, we provide a general introduction to topological spaces. Section 3.2 distinguishes two perspectives in topology: the traditional one, and the computer science one. (Here we're focusing on the latter.) In section 3.3, we discuss how partial orders can be added to topological spaces (like the generalization order that we have already seen). And in section 3.4, we define the particular ordered topological spaces that we will consider: the compact ordered spaces. And we show that they are equivalently described without the order as stably compact spaces. Then we have both the algebraic and the spatial side together, so we can prove the duality result in the next chapter.

3.1 Introduction to topological spaces

When we hear of 'space', we naturally think of the three-dimensional space we live in. And this indeed is an example of a topological space. It is the three-dimensional Euclidean space $\mathbb{R}^3$ whose points $x = (x_1, x_2, x_3)$ are described by the values on the x-axis, the y-axis, and the z-axis. From high-school we also know what lines and planes are in this space, and what their geometry is.

But there also are other spaces. For example, the surface of a sphere. Its points are those (x_1, x_2, x_3) with $x_1^2 + x_2^2 + x_3^2 = 1$. But its geometry is different: for instance, the angles of a triangle add up to more than 180 degrees. Yet another space is the spacetime that we live in according to general relativity. Its points $x = (x_1, x_2, x_3, x_4)$ are four-dimensional—with three spatial and one temporal component—and its geometry is given by a metric tensor. And there are even wilder spaces, where it might not even make sense to speak of a 'geometry' (e.g., angles between lines), but only of 'spatial' properties (e.g., continuous paths from one point to another).

Here we only refer to an intuitive difference between 'geometric' vs 'spatial' (or topological) properties: the latter are invariant under stretching and squishing the space, but the former are not. This is why topology is colloquially also described as rubber sheet geometry.

After much research, mathematicians—most notably Felix Hausdorff in 1914—came up with a general definition of a topological space that includes all these examples. When one first reads this rather abstract definition, one wonders how it possibly can cover all the relevant spatial concepts of the specific examples. But we see how, just from this parsimonious definition of a topological space, we can define many of the common spatial concepts. Again, we split this discussion into objects (topological spaces) and morphisms (continuous functions between spaces).

Mathematics provides many formal notions of space (e.g., Euclidean space, vector space, Hilbert space, probability space, Banach space, etc.). But topological spaces are a very general such notion.

3.1.1 Objects: topological spaces

Without further ado, here is the abstract definition of a topological space.

Definition 3.1. A *topological space* is a pair (X, τ) where X is a nonempty set and τ is a collection of subsets of X such that

Some also allow the empty topological space.

1. $\emptyset$ and X are in τ
2. If $U, V \in \tau$, then $U \cap V \in \tau$
3. If $U_i \in \tau$ is a collection of sets indexed by a set I, then $\bigcup_{i \in I} U_i \in \tau$.

Equivalently: τ is closed under finite intersection and arbitrary union (which includes the empty intersection X and the empty union $\emptyset$). In particular, τ is a sublattice of 2^X.

We also call τ a topology on X. And we call the elements of X *points*. The elements of τ are called *open sets* (or *opens*). Their complements, i.e., sets of the form $C = X \setminus U$ for $U \in \tau$, are called *closed sets*. A subset $K \subseteq X$ that is both open and closed (i.e., $K \in \tau$ and $K^c \in \tau$) is called *clopen*. We just speak of the topological space X if τ is clear from context. Then we write $\Omega(X)$ for the opens of X. The collection of closed (resp. clopen) subsets of X is denoted $\mathcal{C}(X)$ (resp. $\mathsf{Clp}(X)$).

Let's first see that this indeed generalizes our spatial intuitions about 'our' space:

Example 3.2. The three-dimensional space as a topological space: the underlying set is $X := \mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$ and the opens are those subsets $U \subseteq \mathbb{R}^3$ that allow some 'wiggle-room', which is made precise as follows. Recall that the usual distance between two points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

So a subset $U \subseteq \mathbb{R}^3$ is defined to be open precisely if:

1. for all $x \in \mathbb{R}^3$, if $x \in U$, then there is $\epsilon > 0$ such that for all $x' \in \mathbb{R}^3$ with $d(x, x') < \epsilon$, we have $x' \in U$.

This is called the *Euclidean topology* on $\mathbb{R}^3$. ⌟

Another, more abstract example are the two trivial topologies:

Example 3.3. For any nonempty set X, the set $\tau := 2^X$ is a topology on X. It is called the *discrete* topology. Also $\tau := \{\emptyset, X\}$ is a topology on X. It is called the *indiscrete* topology. ⌟

Next, we define some central concepts for a topological space X. They should give a sense of how many concepts one can express with just talk of open sets.

1. *Interior and closure*. If $S \subseteq X$ is a subset, there is a largest open set contained in S, which is called the *interior* of S:

$$\mathsf{Int}(S) := \bigcup \{U \in \tau : U \subseteq S\}.$$

Convince yourself that (a) this is an open set, (b) it is contained in S, and (c) it is the largest such set.

There also is a smallest closed set containing S, which is called the *closure* of S:

$$\mathsf{Cl}(S) := \bigcap \{C \in \mathcal{C}(X) : S \subseteq C\}.$$

Convince yourself that closed sets are closed under arbitrary intersection, so this is indeed a closed set.

2. *Neighborhood*. A subset $S \subseteq X$ is a *neighborhood* of a point $x \in X$ if $x \in \mathsf{Int}(S)$. Accordingly, an *open neighborhood* of a point is an open set containing this point. (If it's clear we're talking about an open neighborhood, we might drop the adjective 'open'.)

3. *Dense*. A subset $S \subseteq X$ is *dense* (in X) if for all points $x \in X$ and open neighborhoods U of x, there is a point $s \in S$ with $s \in U$. So the points of X can be approximated arbitrarily closely by points in S. An equivalent formulation is: $\mathsf{Cl}(S) = X$.

For example, the (countable) set S of all points in $\mathbb{R}^3$ with rational coordinates is dense in $\mathbb{R}^3$.

4. *Convergence*. A sequence $(x_n)_{n\in\mathbb{N}}$ of points in X *converges* to a point $x \in X$ if for all open neighborhoods U of x, there is $N \geqslant 0$ such that, for all $n \geqslant N$, we have $x_n \in U$. We also say that x is the *limit* of the sequence (x_n).

5. *(Sub)base*. Given a nonempty set X, any collection $\mathcal{S}$ of subsets of X *generates* a topology $\langle \mathcal{S} \rangle$: namely, the smallest topology that contains $\mathcal{S}$. This exists because an arbitrary intersection of topologies on X is again a topology on X. Concretely, $\langle \mathcal{S} \rangle$ is the set of arbitrary unions of finite intersections of elements of $\mathcal{S}$.

   If τ is a topology on X, a collection $\mathcal{S}$ of subsets of X is called a *subbase* of τ if $\tau = \langle \mathcal{S} \rangle$. So the opens of τ are arbitrary unions of finite intersection of subbasic elements.

The point (no pun intended) of (sub)bases is to have a more succinct description of the topology. For example, a base for the Euclidean topology on $\mathbb{R}^3$ is given by the open balls $B_\epsilon(x) := \{y \in \mathbb{R}^3 : d(x, y) < \epsilon\}$ for $x \in \mathbb{R}^3$ and $\epsilon > 0$.

Finally, a *base* for the topology τ is a subbase $\mathcal{S}$ such that for every open neighborhood U of any point $x \in U$, there is $V \in \mathcal{S}$ such that $x \in V \subseteq U$.

Equivalently, a base is a collection of open subsets of X such that every open set is a union of of elements from the base.

There is an important classification of topological spaces according to which increasingly stronger so-called *separation axioms* they satisfy. There are many such axioms and they all are of the form that two distinct points can be—in various senses—separated by the topology. The five main ones are the following for a topological space X.

1. X is T_0 (aka *Kolmogorov*) if, for all $x \neq y$ in X, there is an open $U \subseteq X$ such that U contains exactly one of x and y.

2. X is T_1 (aka *Fréchet*) if, for all $x \neq y$ in X, there is an open $U \subseteq X$ such that $x \in U$ and $y \notin U$.

3. X is T_2 (aka *Hausdorff*) if, for all $x \neq y$ in X, there are disjoint opens $U, V \subseteq X$ such that $x \in U$ and $y \in V$.

4. X is T_3 (aka *regular*) if X is T_1 and, for all $x \in X$ and closed $C \subseteq X$ with $x \notin C$, there are disjoint opens $U, V \subseteq X$ with $x \in U$ and $C \subseteq V$.

5. X is T_4 (aka *normal*) if X is T_1 and, for all disjoint closed $C, D \subseteq X$, there are disjoint opens $U, V \subseteq X$ with $C \subseteq U$ and $D \subseteq V$.

Finally, we define the concept of compactness. It formalizes the intuition that the space does not extend infinitely but has finite bounds.

Definition 3.4 (Compactness)**.** Let X be a topological space. If $S \subseteq X$ is a subset, an *open cover* $\mathcal{U}$ of S is a collection of open sets such that $S \subseteq \bigcup_{U \in \mathcal{U}} U$. A subset $S \subseteq X$ is *compact* if every open cover $\mathcal{U}$ of S contains a finite subcover, i.e., there is a finite subset $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\mathcal{U}_0$ is an open cover of S. The space X is called compact if $S := X$ is compact.

For example, while the whole Euclidean space is not compact, closed boxes in it like the unit cube

$$[0,1] \times [0,1] \times [0,1] = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leqslant x_1, x_2, x_3 \leqslant 1\right\}$$

are compact. Also any finite subset of a space is compact. For these and more examples, see exercise 3.c.

A local version of compactness is this: A topological space X is *locally compact* if, for any open neighborhood U of any point $x \in X$, there is an open $V \subseteq X$ and compact $K \subseteq X$ such that $x \in V \subseteq K \subseteq U$. If X is Hausdorff, then compactness implies local compactness, but this is not true in general. And local compactness does also not imply compactness (the Euclidean space is locally compact but not compact).

Two useful results regarding compactness are the following.

1. Finite intersection property characterization. Let X be a topological space and $S \subseteq X$ a subset. A collection $\mathcal{A}$ of closed sets has the *finite intersection property* with respect to S if for every finite subcollection $\mathcal{A}_0$, there is $x \in S$ such that $x \in \bigcap \mathcal{A}_0$. Then S is compact iff, for every collection $\mathcal{A}$ of closed sets with the finite intersection property with respect to S, there is $x \in S$ with $x \in \bigcap \mathcal{A}$.

If $S = X$, we omit the 'with respect to S'.

2. Alexander Subbase Theorem. Let X be a topological space and $\mathcal{S}$ a subbase. If every cover $\mathcal{U} \subseteq \mathcal{S}$ of X has a finite subcover, then X is compact.

The proof of this requires a non-constructive principle, i.e., a version of the axiom of choice. As this is an axiom of standard set theory, we assume this throughout in this course.

3.1.2 Morphisms: continuous functions

Now that we know what topological spaces are, what are the structure-preserving mappings between them? Again, the abstract definition is this.

Definition 3.5. Let X and Y be topological spaces and $f : X \to Y$ a function. We say f is continuous if, for all open subsets V of Y, the preimage $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is an open subset of X.

Example 3.6. As an example to illustrate this definition, consider the usual, so-called epsilon–delta definition of a continuous function $f : \mathbb{R} \to \mathbb{R}$. This definition says that $f : \mathbb{R} \to \mathbb{R}$ is continuous if

1. For every $x \in \mathbb{R}$ and every $\epsilon > 0$, there is $\delta > 0$ such that, for all $y \in \mathbb{R}$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

This captures the idea that, to draw the graph of the function, you do not have to lift your pen: If you want to continue drawing the graph a bit to the left or right of an argument x, the value outputted by the function will not 'jump away' but be close to the value at point x. To illustrate, consider

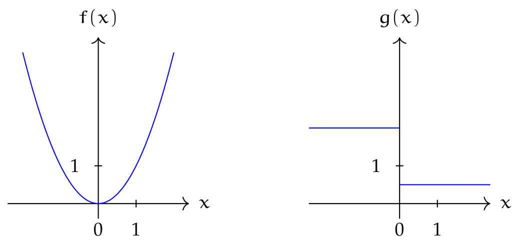


Figure 3.1: A continuous function f (left) and a non-continuous function g (right).

the following two functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := x^2 \qquad\qquad f(x) := \begin{cases} 2 & \text{if } x < 0 \\ 0.5 & \text{if } x \geqslant 0. \end{cases}$$

When drawing their graphs, as in figure 3.1, we can do this for f without lifting the pen, while for g we have to lift it at $x = 0$. And indeed, for $\epsilon := \frac{1}{4} > 0$, we cannot find the required $\delta > 0$.

Verify this for yourself.

It is a good exercise (see exercise 3.d) to show that this 'hands-on' definition of continuity is equivalent to—and hence generalized by—the abstract topological definition. For this, we have to define the standard topology on the real line $\mathbb{R}$. This is done just like in the three-dimensional case, except that the distance function now simplifies: Here, since $\mathbb{R}$ has just one dimension, $d(x, y) = \sqrt{(x-y)^2} = |x - y|$. So the opens of the real line are those subsets $U \subseteq \mathbb{R}$ such that, for all $x \in \mathbb{R}$, if $x \in U$, then there is $\epsilon > 0$ such that, for all $x' \in \mathbb{R}$ with $d(x, x') < \epsilon$, we have $x' \in U$. ⌟

Okay, this time the pun is intended

Some further useful terminology around continuous functions is the following. A continuous function $f : X \to Y$ between topological spaces is

- *open* if, for all open $U \subseteq X$, the image $f[U] = \{f(x) : x \in U\}$ is an open subset of Y.
- *closed* if, for all closed $C \subseteq X$, the image $f[C] = \{f(x) : x \in C\}$ is a closed subset of Y.
- a *homeomorphism*, if f has a continuous inverse, i.e., f is a bijection and both f and f^{-1} are continuous. (Equivalently, as exercise 3.e shows, f is a continuous and open bijection; this is further equivalent to f being a continuous and closed bijection.)

Note the additional 'e': it is not 'homomorphism' as with lattices.

- an *embedding*, f is injective and, for each open $U \subseteq X$, there is an open $V \subseteq Y$ such that $f[U] = f[X] \cap V$. Equivalently, the function $f : X \to f[X]$ is a homeomorphism when giving $f[X] \subseteq Y$ the subspace topology (whose opens are $V \cap f[X]$ for $V \subseteq Y$ open).

This is the conceptual meaning of embedding: X is, up to homeomorphism, a subspace of Y.

Homeomorphisms are the isomorphisms of spaces: If there is a homeomorphism between spaces they are classed homeomorphic and hence are topologically the same. The standard example is that a donut and a coffee mug are homeomorphic: you can obtain one from the other by squishing and squeezing, but—importantly—without breaking and tearing.

Hence the common joke that topologists cannot tell them apart.

3.1.3 Constructions: subspaces, products, quotients

Coming to constructions with topological spaces, we have the following.

Cf. the trinity of sublattice, product, and homomorphic images/quotient for lattices.

1. Subspace. Given a topological space (X, τ), any nonempty subset $Y \subseteq X$ can be naturally made into a topological space by equipping it with the *subspace topology*

$$\tau \upharpoonright Y := \{U \cap Y : U \in \tau\}.$$

2. Product topology. If $(X_i)_{i \in I}$ is a collection of topological spaces indexed by a set I, the product space $\prod_{i \in I} X_i$ has as underlying set the Cartesian product of the sets X_i and its topology is generated by the subbase of sets of the form $\left\{x \in \prod_{i \in I} X_i : x_j \in V\right\}$ for $j \in I$ and $V \subseteq X_j$ open. Tychonoff's Theorem says that the arbitrary product of compact spaces is again compact.

Equivalently, this is the smallest topology making continuous all the projections $\pi_i : \prod_I X_i \to X_i$ mapping x to its i-th component x_i.

3. Quotient space. If X is a topological space and $\equiv$ an equivalence relation on X, the quotient space has as underlying set $X/\equiv$ and the opens are those sets $U \subseteq X/\equiv$ such that $\{x \in X : [x]_\equiv \in U\}$ is open in X.

A construction specific to spaces is that we can take the *join* of two topologies that live on the same underlying set. This is made precise as follows.

1. If X is a nonempty set, then

$$\mathrm{Top}(X) := \left\{\tau \in 2^{2^X} : \tau \text{ is a topology}\right\}$$

is, when ordered under inclusion, a complete lattice.

2. Infima are given by intersections, and suprema are given by the topology generated by unions. The least element is the indiscrete topology, and the greatest element is the discrete topology.

3. In particular, if σ and τ are two topologies on X, then their *join* $\sigma \vee \tau$ is the topology generated by $\sigma \cup \tau = \{U \subseteq X : U \in \tau \text{ or } U \in \sigma\}$.

Some fun facts are that the Hausdorff topologies on X form an upset in $\mathrm{Top}(X)$, and the compact topologies form a downset. And compact-Hausdorff topologies are incomparable: If σ is a Hausdorff topology and τ a compact topology on the nonempty set X, then $\sigma \subseteq \tau$ implies $\sigma = \tau$.

A proof of this can be found in Gehrke and van Gool (2023, Prop. 2.10).

3.2 Two perspectives on topological spaces

After this formal discussion of topological spaces, we now take a moment to conceptually reflect on them. The point we make is that the abstract definition of a topological space actually unites two different perspectives on spaces: the traditional one and the computer science one—and they differ in the Hausdorff separation axiom.

For a textbook treatment of this, see the first pages of Vickers (1989). For an influential paper, see Smyth (1983).

3.2.1 The traditional perspective (Hausdorff)

On the traditional perspective on topological spaces, we look at our three-dimensional space or generalizations thereof: i.e., spaces that still conform to our three-dimensional intuition. In particular, they are Hausdorff: It is fundamental to our conception of space that we can encapsulate two distinct points in two disjoint open balls, however small. The spaces studied under this perspective are the traditional ones found in calculus and geometry, and they are almost always Hausdorff.

An exception are the spectra of rings with their Zariski topology.

We showed how to generalize the idea of an open set arises from the intuition of a set with wiggle-room. These sets satisfy the axioms of a topological space, but this motivation does make clear why to choose *exactly* these axioms and not further ones. Indeed, it is not clear why not to also demand the Hausdorff separation axiom.

That there is a clear motivation also for the Non-Hausdorff case became apparent via computer science with the discovery of domain theory in the late 1960s.

3.2.2 The computer science perspective (Non-Hausdorff)

On the computer science perspective on topological spaces, we think of the open sets as the properties and we think of the points as the things that can have these properties. And we require the properties to be observable: if a thing has a property, we should be able to make a measurement which confirms this after a finite amount of time.

This perspective was already at play in our intuitive motivation of a duality: that between objects and their properties (section 1.1.1). Rephrased in terms of topological spaces, we say that X is the set of objects under consideration, and τ is the set of (extensions of) properties of these objects. So to say that object $x \in X$ has property $U \in \tau$ is to say $x \in U$. (In logical notation, we might also write $x \vDash U$.)

If τ is the whole powerset of X—i.e., the discrete topology—, we count any subset of X as an extension of a property under consideration. But since we want to consider observable properties, this need not always be the case. For example, the property U of *weighing exactly* 2 kg is not observable. This is because if an object x has U, we cannot confirm this through a measurement, because any scale that we use will have a margin of error $\epsilon > 0$. Concretely, there can be an object x' that weighs $\frac{\epsilon}{2}$ kg more than x, so it does not have U, but we cannot detect this with our scale. So the problem is that U did not have any wiggle-room to accommodate margins of error in measurements.

Moreover, and moving closer to computer science, the things need not be 'complete' objects in our world like chairs and trees. They could also be 'partial' objects like the (interim-) outputs generated by computational processes. As a simple example, consider a process that produces better and better approximations of the number π. So X is the set of these approximations: i.e., 3, 3.1, 3.14, 3.141, 3.1415, etc., and we might add π as the complete infinite output that is produced in the limit. An observable property U is, for example, being precise up to the third digit, so $U = \{3.141, 3.1415, \ldots\}$. In general, an observable property is a subset $U \subseteq X$ such that, if an approximation x has U, we can find a proof of this, so all more precise approximations also have U. This space is not Hausdorff, because even if $3.14 \neq 3.141$, there cannot be two disjoint open sets U and V with $3.14 \in U$ and $3.141 \in V$, because once 3.14 is in U, also the more informative 3.141 is in U, so $U \cap V$ is nonempty.

A computer scientist would say: X is a data type with the open sets being the semidecidable properties on the type (Smyth 1983, p. 664).

On this perspective, the axioms of a topological space precisely describe the observational character of the properties: if true, we will find out via observation after a finite amount of time. And these properties are precisely closed under finite conjunction (running the measurements for the conjuncts in sequence) and under arbitrary disjunction (running the measurements for the disjuncts in parallel).

If 'observable' is understood as 'semidecidable' (as above), we need to qualify 'arbitrary' as 'over an effective index set' (or presuppose an effective mathematical universe).

To summarize, on the computer science perspective, we view the topology as describing the observations we can make about the points in the space. The defining axioms of topological spaces are the general laws governing the observations.

Remark 3.7. Now you might ask yourself the question: The observations hence also are governed by a logic, albeit an infinitary one. So isn't this already our duality? The connection from space to logic? The short answer is: no, but its on the right track. The long answer is this. The logic with finite conjunctions and arbitrary disjunctions is known as *geometric logic*, and it indeed has been identified as the logic of observations (Abramsky 1991, p. 16). But for a duality we would want a description of it in terms of properties/propositions alone, without reference to objects/models having them/making them true. This is possible. (The key words is 'pointless topology' or 'locale theory' and the algebraic object describing such sets of propositions are known as frames: you can read up on this in Vickers (1989).) But, moreover, we would also like the spaces to correspond to the usual *finite* logics with finite algebraic operations. The key insight for Priestley duality will be that we don't need to take *all* possible observations for the logic: there is a subset of 'finitary' observations whose logic already determines all the observations. And their structure is precisely determined by distributive lattices. But this is what we still need to work toward.

This remark might be more confusing than helpful: if you didn't ask yourself this question, maybe just skip it.

3.3 Orders on topological spaces

We can move from orders to spaces in two directions. If we start with a space, we have already seen that we can naturally define an order on that space: the generalization order—or, more common, its opposite, the specialization order (section 3.3.1). If we start with a partial order, there are also several ways to define a topology on it based on that order (section 3.3.2).

3.3.1 The specialization order

In the finite duality (specifically, at the end of section 2.4.1), we already encountered the idea of generalization: an object y is more general than an object x is any property (among those properties under consideration) that y has also x has. The opposite order is that of *specialization*: x is more special than y if any property that x has also y has. As these are just opposites orders, they are essentially equivalent, but it is the specialization order that is usually considered for topological spaces.

Definition 3.8. Let X be a topological space. The *specialization order* $\leqslant_X$ is defined by

$$x \leqslant_X y :\Leftrightarrow \forall U \in \Omega(X) : x \in U \Rightarrow y \in U.$$

A subset of X is called *saturated* if it is an upset in the specialization order. We write $\mathsf{KS}(X)$ for the collection of subsets of X that are both compact and saturated (aka *compact-saturated* subsets).

The following collects some basic facts, the proof of which is exercise 3.g.

Proposition 3.9. *Let X be a topological space and $\leqslant_X$ its specialization order.*

1. *$\leqslant_X$ is a preorder.*
2. *For $x \in X$, we have $\downarrow x = \mathsf{Cl}(\{x\})$.*
3. *$\leqslant_X$ is a partial order iff X is T_0.*
4. *$\leqslant_X$ is the identity relation iff X is T_1.*
5. *A subset $S \subseteq X$ is saturated iff it is an intersection of open sets.*
6. *A subset $S \subseteq X$ is compact iff its saturation $\uparrow S$ is compact.*
7. *If X is compact and Hausdorff, then $\mathsf{KS}(X) = \mathcal{C}(X)$.*

Some comments:

1. Looking at the specialization order is particularly natural from the computer science perspective discussed in section 3.2.2. If the objects are not necessarily 'complete' but only 'partial', it especially makes sense to say that one object y is more special than an object x in the sense that y 'is more determined' or 'has more information' than x.
2. In the Hausdorff setting (which implies T_1), the specialization order is trivial (as item 4 shows). But in the Non-Hausdorff setting central to computer science, the specialization order carries a lot of information.
3. The last item shows that in compact Hausdorff spaces the compact-saturated subsets coincide with the closed subsets. However, from the logical perspective, the spaces usually are compact but fail to be

Hausdorff (unless one starts from a Boolean algebra). In that case, it's a good trick to remember to move from the closed subsets to the more general compact-saturated subsets. We see this in action in definition 3.11 below.

This also becomes relevant, e.g., in the construction of the upper Vietoris space/Smyth powerdomain (a topological analogue of the powerset). See Gehrke and van Gool (2023, Def. 6.21).

4. A similar move is to go from clopen subsets to compact-open subsets. (This is how Stone's duality is generalized from Boolean algebras to distributive lattices, and Priesley's duality takes another, but equivalent route.)

3.3.2 Order topologies

If we start with a partial order $(X, \leqslant)$, there are various ways we can put a topology on X. Naturally we first look for those topologies on X whose specialization order is $\leqslant$. Here are three:

1. The *upper topology* on X is the least topology in which $\downarrow x$ is closed for every $x \in X$:

   $$\iota^{\uparrow}(X) := \langle \{(\downarrow x)^c : x \in X\} \rangle$$

   Recall that $\langle \mathcal{S} \rangle$ is the topology generated by $\mathcal{S}$.

   It is the least/coarsest topology on X whose specialization order is $\leqslant$.

2. The *Scott topology* on X, written $\sigma(X)$, has as open sets precisely the subsets $U \subseteq X$ with the following property: U is an upset and for every directed subset $D \subseteq X$, if $\bigvee D$ exists and $\bigvee D \in U$, then there is $d \in D$ with $d \in U$. In words, the opens of σ are precisely the upsets of X that are inaccessible by directed joins.

   Recall that $D \subseteq X$ is directed if D is nonempty and, for all $a, b \in D$, there is $c \in D$ with $a, b \leqslant c$.

   This gives rise to the field of *domain theory* (Abramsky and Jung 1994). It studies directed-complete partial orders (aka dcpo's), i.e., partial orders where every directed subset has a least upper bound. The theory has been developed to give semantics to programming languages. The key idea is that the function specified by a while loop (or recursion) is the join of the directed set of finite iterations of the while loop.

3. The *Alexandrov topology* on X is

   $$\alpha(X) := \{U \subseteq X : U \text{ is an upset}\}$$

   It is the largest/finest topology on X whose specialization order is $\leqslant$.

Often it also useful to consider the opposite order. For example,

4. The *lower topology* on X is

$$\iota^{\downarrow}(X) := \langle\{(\uparrow x)^c : x \in X\}\rangle$$

5. The *dual Alexandrov topology* on X has as opens the downsets of X.

Finally, we can also combine these topologies by forming joins.

6. The *interval topology* on X is the join of the upper and lower topologies:

$$\iota(X) := \iota^{\uparrow}(X) \vee \iota^{\downarrow}(X).$$

   The usual topology on the real line is the interval topology given by the usual order on it.

7. The Lawson topology on X is the join of the Scott and the lower topology. After the Scott topology, it arguably is the second most important topology in domain theory.

8. The join of the Alexandrov and the dual Alexandrov topology is the discrete topology.

3.4 Priestley spaces and spectral spaces

In this section, we define the two types of spaces that we will encounter on the spatial side of the duality. This is because we will consider two ways of associating distributive lattices with spaces: Priestley duality and Stone duality. Subsection 3.4.1 will introduce the spaces used in the first duality and subsection 3.4.2 those of the second. The two dualities—and hence the spaces—are very closely related. How exactly, will be made precise in the next chapter.

3.4.1 Priestley spaces

Intuitively, Priestley spaces are compact topological spaces together with a partial order that interacts nicely with the topology.

Definition 3.10. An *ordered space* is a triple $(X, \tau, \leqslant)$ such that

1. (X, τ) is a topological space,

2. $(X, \leqslant)$ is a partial order, and

3. $\leqslant \subseteq X \times X$ is closed in the product topology.

We further explain this requirement below the definition.

An ordered space is a *compact ordered space* if the underlying topological space is compact.

A *Priestley space* is a compact ordered space that also satisfies the following separation axiom known as being *totally order-disconnected* (TOD):

> For all $x \nleqslant y$ in X, there is a clopen downset U in X such that $y \in U$ and $x \notin U$.

In exercise 3.h, you show that the TOD property already implies (3), so we can and do remove (3) from the definition of a Priestley space.

A *morphism* from an ordered space to another is a function on the underlying sets that is both continuous with respect to the topologies and monotone with respect to the orders. An *isomorphism* (or order-homeomorphism) is a morphism that is both a homeomorphism with respect to the topologies and an order-isomorphism with respect to the orders.

We consider an example below (example 3.12). One comment on condition (3): Why should we require this for a 'nice interaction' between the order and the topology? As an analogy, consider the graph of a function $f : X \to Y$ between two topological spaces, which is a subset of $X \times Y$. (Formally, a function is even identified with its graph.) To say that the function 'nicely interacts' with the topology is to say that it is continuous, which implies, if Y is Hausdorff, that the graph is closed (closed graph theorem). Condition (3) generalized this intuition to the subset $\leqslant \subseteq X \times X$. Moreover, exercise 3.h provides an equivalent formulation of condition (3) as the natural reformulation of the Hausdorff separation axiom in the order context.

3.4.2 Spectral spaces

Intuitively, stably compact spaces are compact spaces whose compact-saturated subsets are reasonably stable under intersections, and they are spectral if they additionally have 'nice' bases.

Definition 3.11. A stably compact space is a topological space X with the following properties:

1. X is T_0

2. X is compact

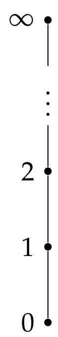


Figure 3.2: The order $\omega + 1$.

3. X is locally compact

4. X is *coherent*, i.e., the intersection of any two compact-saturated subsets is again compact (it automatically is saturated again).

5. X is *well-filtered*, i.e., for any filtered collection $\mathcal{F}$ of compact-saturated subsets of X and any open set U of X, we have that, if $\bigcap \mathcal{F} \subseteq U$, then there is $K \in \mathcal{F}$ such that $K \subseteq U$.

Recall that a subset A of a poset P is filtered if, for any $a, b \in A$, there is $c \in A$ with $c \leqslant a, b$.

It is called a *spectral space* if it additionally satisfies

6. X has a base of compact-opens.

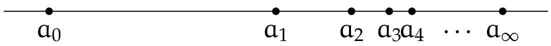
A *spectral map* $f : X \to Y$ is a continuous map between spectral spaces such that the preimages of compact-open sets are compact.

Priestley spaces and spectral spaces are closely related: one can translate one into the other. How to do this precisely, we will see in the next chapter. Let's end with an example.

Example 3.12. Consider the set $\omega + 1 = \{0, 1, 2, \ldots, \infty\}$ with the order $\leqslant$ where the natural numbers are ordered in the usual way and ∞ is bigger than all natural numbers. This is depicted in figure 3.2. As a topology on $\omega + 1$, consider the so-called (Alexandroff) one-point compactification of the natural numbers $\mathbb{N}$ equipped with the discrete topology. Concretely, this means the open sets of $\omega + 1$ are given by

$$\tau := \big\{A : A \subseteq \mathbb{N}\big\} \cup \big\{(\mathbb{N} \setminus F) \cup \{\infty\} : F \subseteq \mathbb{N} \text{ finite}\big\}.$$

More visually, you can also think of it as the subspace of the real line consisting of those points $a_n = 1 - \frac{1}{n+1}$ (for $n = 0, 1, \ldots$) together with their limit point $a_\infty = 1$ (with their inherited order).



Also consider the topology

$$\tau' := \{U \in \tau : U \text{ is an } \leqslant\text{-downset}\}.$$

Exercise 3.i asks you to prove that then $(\omega + 1, \tau, \leqslant)$ is a Priestley space and $(\omega + 1, \tau')$ is a spectral space. ⌟

3.5 Exercises

Exercise 3.a. To get familiar with the abstract topological concepts from section 3.1.1, we apply them to the usual three-dimensional space $\mathbb{R}^3$ (whose open sets are those with 'wiggle-room').

1. Prove that the collection τ of sets $U \subseteq \mathbb{R}^3$ with wiggle-room, as defined in example 3.2 (1), indeed forms a topology on $\mathbb{R}^3$.

2. Show that a base for this topology is indeed given by the open balls $B_\epsilon(x)$.

3. Show that the unit cube $[0, 1] \times [0, 1] \times [0, 1]$ is closed and that is interior is the open unit cube $(0, 1) \times (0, 1) \times (0, 1)$.

4. Show that the rational points, i.e., those $x = (x_1, x_2, x_3)$ where x_1, x_2, and x_3 are rational numbers, form a dense subset of $\mathbb{R}^3$.

5. Show that the sequence of points $(\frac{1}{n}, \frac{1}{n}, \frac{1}{n})_n \geqslant 1$ converges to $(0, 0, 0)$.

6. Show that $\mathbb{R}^3$ is Hausdorff.

Exercise 3.b. Let X be a topological space. Prove that the interior map $\mathsf{Int} : 2^X \to \Omega(X)$ is upper adjoint to the inclusion map $\iota : \Omega(X) \to 2^X$, and that the closure map $\mathsf{Cl} : 2^X \to \mathcal{C}(X)$ is lower adjoint to the inclusion $\iota' : \mathcal{C}(X) \to 2^X$.

Exercise 2.1.3 in Gehrke and van Gool (2023)

Exercise 3.c. This exercise gets you acquainted with the concept of compactness via some examples.

1. Show that $\mathbb{R}^3$ is not compact but the unit cube is.

2. Show that any finite subset of any topological space is compact.

3. Equip $2 := \{0, 1\}$ with the discrete topology. Define $X := 2^{\mathbb{N}}$ to be the product space $\prod_{\mathbb{N}} 2$ of $\mathbb{N}$-many copies of 2. This is known as the *Cantor space*. Show that it is compact (and don't use Tychonoff's theorem for this).

Exercise 3.d. Show that for a function $f : \mathbb{R} \to \mathbb{R}$ the topological sense of continuity from definition 3.5 is equivalent to the epsilon–delta definition of continuity from example 3.6.

Exercise 2.1.1 (c) in Gehrke and van Gool (2023)

Exercise 3.e. Let $f : X \to Y$ be a continuous bijection. Show that the following are equivalent.

Exercise 2.1.4 (c) in Gehrke and van Gool (2023)

1. f is a homeomorphism (i.e., its inverse is continuous)
2. f is open (i.e., maps open sets to open sets)
3. f is closed (i.e., maps closed sets to closed sets).

Exercise 3.f. This exercise asks you to prove the following useful facts about compact spaces.

1. A closed subset of a compact space is compact.
2. A compact subset of a Hausdorff space is closed. This need not be true without the Hausdorffness assumption.
3. The image of a compact subset under a continuous function is compact.
4. Conclude that a continuous function from a compact space to a Hausdorff space is closed.
5. Conclude with exercise 3.e that a continuous bijection between compact Hausdorff spaces is a homeomorphism.

Exercise 3.g. Prove proposition 3.9.

Exercise 3.h. Let (X, τ) be a topological space and $\leqslant$ a partial order on X. Then the following are equivalent:

1. $\leqslant$ is a closed subset of $X \times X$ (with respect to the product topology)
2. For every $x \nleqslant y$ in X, there are open subsets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $\uparrow U \cap \downarrow V = \emptyset$.

Recall that $\uparrow U = \{x \in X : \exists u \in U. x \geqslant u\}$ *and* $\downarrow V = \{x \in X : \exists v \in V. x \leqslant v\}$.

Conclude that if additionally (X, τ) is compact and satisfies the TOD property with respect to $\leqslant$, then $(X, \tau, \leqslant)$ is a Priestley space. In other words, in the definition of a Priestley space, we can delete the condition that $\leqslant$ is closed.

Exercise 3.i. In example 3.12, we defined the two spaces $(\omega + 1, \tau, \leqslant)$ and $(\omega + 1, \tau')$. For this exercise, you can assume that τ and τ' are topologies (i.e., you don't have to prove this).

1. Show that $(\omega + 1, \tau, \leqslant)$ is a Priestley space.

2. Show that $(\omega + 1, \tau')$ is a spectral space.

Claim (2) will follow from claim (1) via the theorem from the next chapter that relates Priestley spaces and spectral spaces. But you can also show it directly here.

4 Two sides of the same coin: Priestley and Stone duality

In this chapter, we finally prove the duality theory result that we have been working toward. In fact, we show two closely related ones: Priestley and Stone duality. Historically, Stone duality came first. It relates every distributive lattice to a spectral space and vice versa (these topological spaces were defined in definition 3.11). Priestley duality relates every distributive lattice to a Priestley space and vice versa (these ordered topological spaces were defined in definition 3.10). We will see that these two types of spaces are closely related—in fact, they are isomorphic as categories. If we restrict us to Boolean algebras, the two dualities restrict to one and the same duality between Boolean algebras and so-called Stone spaces—this is the duality that Stone is most famous for. All these results are informally summarized in figure 4.1. The formal version of this diagram—which is the summary of the results of this chapter, and in fact the whole course—is in figure 4.3 below.

In section 2.4, we have seen the finite case of the duality result.

In section 4.1, we carefully motivate and state Priestley duality and then prove it in section 4.2. In section 4.3, we state the Stone duality and how it relates to Priestley duality. Finally, in section 4.4, we deal with the special case of Boolean algebras.

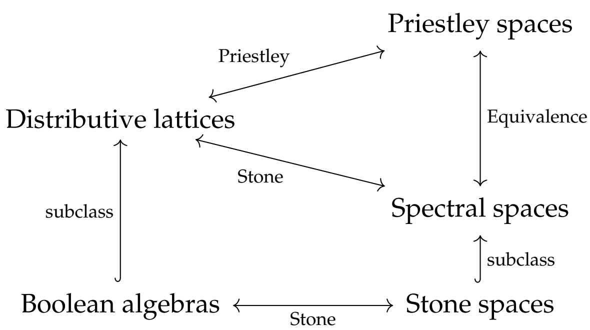


Figure 4.1: The Stone and Priestley duality informally

4.1 Priestley duality

In this section, we motivate and formally state the Priestley duality in a series of propositions. So the Priestley duality formally is the conjunction of propositions 4.3–4.9 below. Category-theoretically this is swiftly expressed as: The category of distributive lattices and lattice homomorphisms is dually equivalent to the category of Priestley spaces and order-preserving continuous functions. In the next section, we will prove these propositions.

4.1.1 From distributive lattices to Priestley spaces

Let's recall again the idea of how to recover a space from a distributive lattice (from chapter 1 and section 2.4.1). We do this in terms of the objects–properties example of a duality, but you can swap this to your favorite example. If we have a distributive lattice L of properties, we can recover the objects as 'decisive' subsets F of L, i.e., those which have the expected closure conditions for implication ($\leqslant$), conjunction ($\wedge$), disjunction ($\vee$), logical truth ($\top$), and logical falsity ($\bot$). We have already called these subsets prime filers:

By 'recover' we mean 'retrieve' or 'regain': Assume we have the lattice but lost the space, then how can we deduce how the space must have been like only using the information provided by the lattice?

Definition 4.1. Let L be a distributive lattice. A subset F of L is a filter if it is a nonempty upset that is closed under meet. It is proper, if $F \neq L$ (equivalently, $\bot \notin F$). A *prime filter* is a proper filter such that, for all $a, b \in L$, if $a \vee b \in F$, then either $a \in F$ or $b \in F$.

Being nonempty is, for an upset, equivalent to $\top \in F$.

The order-dual notion of a filter is sometimes also useful and is called an *ideal*: Conceretely, these are nonempty downsets $I \subseteq L$ closed under join (if $a, b \in I$, then $a \vee b \in I$). An ideal is *proper* if $I \neq L$ and *prime* if, additionally, $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

The notion of an ideal is also important in the theory of rings which is central in commutative algebra. They are used to build the spectra *of rings, which provide useful topological tools to understand a ring. They form spectral spaces—hence the name. To see how lattice ideals relate to ring ideas, see exercise 3.1.9 of Gehrke and van Gool 2023.*

Here we'll always work with filters and rarely mention ideals (only if we need to also talk about duals of filters). But, again, ultimately this is a convention and many textbooks primarily use ideals.

An equivalent way to define a prime filter F of a distributive lattice L is by requiring that its characteristic function

$$\chi_F : L \to \mathbf{2}$$
$$a \mapsto \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases}$$

is a lattice homomorphism. This brings out maybe more clearly the intuition that a prime filter decides, for every property/proposition in L, whether it is true according to it. Although these two characterizations (subsets vs homomorphisms) are equivalent, it often is conceptually useful to consider both. Exercise 4.a asks you to prove this equivalence and a third one that prime filters are those subsets whose complements are prime ideals.

So we recover a space of objects X as the set of prime filters of L. Again, we can order these recovered objects by generalization: G is more general than F (written $F \leqslant G$) iff every property that G has, also F has, but F might have more properties. In simpler terms, G is a subset of F.

As noted, the dual of generalization is specialization, so it ultimately is a matter of convention which we pick. In fact both are used in duality theory, so one should always check which convention is used.

Definition 4.2. If L is a distributive lattice, write $\mathsf{PrFilt}(L)$ for the set of all prime filters of L, ordered by reverse inclusion: $F \leqslant G$ iff $F \supseteq G$. (Similarly, we write $\mathsf{Filt}(L)$ for the set of all filters of L ordered by reverse inclusion.)

What we couldn't yet see so far is that this space of recovered objects not just has a generalization order (like in the finite case) but also really is a space in the sense of topology! So we need to say what the open sets are on $\mathsf{PrFilt}(L)$. But what's a natural choice? The general intuition for the open sets is that they represent different degrees of closeness or similarity: For example, in our space, the open ball $B_\epsilon(x)$ around a point x represents closeness to degree ϵ. What would be basic degrees of similarity for prime filters? The idea is that two prime filters can be close to degree $a \in L$ by agreeing on the property $a \in L$, i.e., both contain a or both do not contain a. So we declare open, for each $a \in L$, the two sets

$$\{F \in \mathsf{PrFilt}(L) : a \in F\} \qquad\qquad \{F \in \mathsf{PrFilt}(L) : a \notin F\}.$$

This indeed in fact produces a Priestley space as the next propositions shows.

Proposition 4.3. *Let L be a distributive lattice. Equip $X := \mathsf{PrFilt}(L)$ with the topology generated by the subbase $\mathcal{S}$ consisting of the following sets, for each $a \in L$,*

$$\widehat{a} := \{F \in X : a \in F\} \qquad \textit{and} \qquad \widehat{a}^c = \{F \in X : a \notin F\}.$$

The X with this topology and the generalization order $\leqslant$ (i.e., reverse inclusion of prime filters) is a Priestley space. It is also denoted $\mathsf{Pr}(L)$.

In this and the next section, we often keep writing $\mathsf{PrFilt}(L)$ for $\mathrm{Pr}(L)$ to remind us that we take prime filters.

Example 4.4. Consider the diamond distributive lattice with two incomparable elements a and b between the bottom element $\bot$ and the top element $\top$ (cf. figure 2.1). What are its prime filters? To compute them by brute force, let's first list all its upsets (since every prime filter in particular is an upset):

$$\emptyset, \{\top\}, \{a, \top\}, \{b, \top\}, \{a, b, \top\}, \{\bot, a, b, \top\}.$$

Which of those are filters? Not the empty set, because filters are required to be nonempty. And also not the second-last set, because it is not closed under $\wedge$. But the remaining sets are filters. The last one is, by definition, not proper. Which of the remaining proper filters are prime? The filter $\{\top\}$ is not prime, because $a \vee b = \top$, but neither a nor b are in the filter. But the two filters

$$F_1 := \{a, \top\} \qquad\qquad F_2 := \{b, \top\}$$

are prime. So they form the points of the dual space $X = \{F_1, F_2\}$.

What is the order on X? Recall that this is given by the reverse inclusion order on filters. However, neither F_1 is a subset of F_2 nor vice versa. So in this case the order $\leqslant$ is just the identity relation: there is no nontrivial generalization between objects.

Finally, what is the topology of the space X? We in particular have the open sets:

$$\begin{aligned}
\widehat{\bot} &= \{F \in X : \bot \in F\} = \emptyset \\
\widehat{a} &= \{F \in X : a \in F\} = \{F_1\} \\
\widehat{b} &= \{F \in X : b \in F\} = \{F_2\} \\
\widehat{\top} &= \{F \in X : \top \in F\} = \{F_1, F_2\}
\end{aligned}$$

These in fact already are all the subsets of the space X, to the topology is discrete in this case. ⌟

4.1.2 From Priestley spaces to distributive lattices

Now going in the other direction, if we have a space of objects, what are the properties of these objects? We again identify a property with its extension: i.e., the set of objects having the property. So properties are subsets of the space. But they are not just *any* subset.

We already saw that extensions are downsets with respect to the generalization order: If object y is more general than object x, i.e., $x \leqslant y$, and if y has property a, i.e., y is in (the extension of) a, then the more special object x also has property a, i.e., is in a.

What we couldn't see before, but what comes to light through the topology on the space, is that extensions should be clopen sets. We already saw this above for the recovered objects: If we have a lattice of properties L, then for each property $a \in L$, its extension in the recovered space X is $\widehat{a} = \{F \in X : a \in F\}$, and by construction this $\widehat{a}$ is a clopen set. And it stands to reason that this should also be the case if we start with a space of objects X. Indeed, it seems plausible to require that the extension of a property a is closed under similarity: if object x has property a and object y is very similar to x, then also y has property a. If similarity is spelled out topologically, this requirement naturally is formalized as the extension of a being clopen: if x is in the extension, there is a degree of similarity (i.e., an open set) such that all objects similar to x by at least this degree also are in the extension (hence the extension is open); and if x is not in the extension, there is a degree of similarity (i.e., an open set) such that all objects similar to x by at least this degree also are not in the extension (hence the extension is closed).

For much more on this, see, e.g., Belastegui Lazcano (2020).

So the properties of a space of objects—identified with their extensions—are clopen downsets of the space. Fortunately, they form a distributive lattice.

Proposition 4.5. *Let* $(X, \tau, \leqslant)$ *be a Priestley space. Let* $L := \mathsf{ClpD}(X)$ *be the set of clopen downsets of* X *ordered by inclusion. Then* L *is a distributive lattice.*

Example 4.6. Let's consider again the Priestley space $X = \omega + 1$ from example 3.12. Its points were $X = \{0, 1, 2, \ldots, \infty\}$ with the expected order, and its topology was the one-point compactification. So what are the clopen downsets?

The open sets $U \subseteq X$ are either of the form $U = A$ for $A \subseteq \mathbb{N}$ or of the form $U = (\mathbb{N} \setminus F) \cup \{\infty\}$ for $F \subseteq \mathbb{N}$ finite.

When are these also downsets? For the first form, this is the case if for any $n \in A$, all $m \leqslant n$ also are in A. For the second form, this only is the case if $U = X$: because since $\infty \in U$, also all lower elements need to be in U, which is all elements.

And when are these open downsets also closed? If $U = X$, then U automatically is closed (the complement $\emptyset$ is always open). If $U = A \subseteq \mathbb{N}$, then this is closed iff the complement $(\mathbb{N} \setminus A) \cup \{\infty\}$ is open, which is the case precisely if A is finite.

So the clopen downsets are precisely those of the form $U_n = \{0, 1, \ldots, n\}$ for some $n \in \mathbb{N}$ and $U_\infty = \{0, 1, \ldots, \infty\}$, together with the empty set. If we order those by inclusion, we get what's depicted in figure 4.2. So we

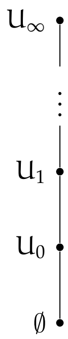


Figure 4.2: The distributive lattice $\mathsf{ClpD}(\omega + 1)$.

have the very special case that the space with its partial order is actually isomorphic to its dual distributive lattice! ⌟

4.1.3 Lattices are isomorphic to their double-duals

We just mentioned the idea that if we start with a lattice of properties L, we can map each property $a \in L$ to its extension $\widehat{a}$ in the space of recovered objects. To be justified in identifying properties with their extensions, we should expect that this map $a \mapsto \widehat{a}$ in fact is an isomorphism! Fortunately, this is the case.

Proposition 4.7. *Let* L *be a distributive lattice. Then the following is a well-defined lattice isomorphism between* L *and its double-dual:*

$$\begin{aligned} \widehat{\cdot} \colon L &\to \mathsf{ClpD}(\mathsf{PrFilt}(L)) \\ a &\mapsto \widehat{a} = \{F \in \mathsf{PrFilt}(L) : a \in F\}. \end{aligned}$$

4.1.4 Spaces are isomorphic to their double-duals

Similarly, we also expect that the space of objects is isomorphic to its a double-dual. In section 1.1.1, we discussed the motivation behind this:

- Each object x determines a set F_x of properties consisting of precisely those properties that x has—and this is a prime filter.
- Each set F of properties that is a prime filer should determine an object x: namely, the object having precisely the properties in F.

This indeed yields an isomorphism between a space and its double-dual:

Proposition 4.8. $(X, \tau, \leqslant)$ *be a Priestley space. Then the following is a well-defined order homeomorphism between* X *and its double-dual:*

$$\beta : X \to \mathsf{PrFilt}(\mathsf{ClpD}(X))$$
$$x \mapsto \{a \in \mathsf{ClpD}(X) : x \in a\}$$

the greatest x *with* $x \in a$ *for all* $a \in F \leftarrow F$

4.1.5 Also including morphisms

Finally, we also want to relate morphisms on the algebraic side with those on the spatial side. So far, we have related lattices to spaces (i.e., the objects of the respective categories), but we also want to relate connections between lattices to connections between spaces (i.e., the morphisms between the respective categories).

More concretely, and analogously to the finite case, so far we have an exact correspondence between Priestley spaces X and distributive lattices L, by relating X to $\mathsf{ClpD}(X)$ and L to $\mathsf{PrFilt}(L)$. The previous two subsections showed that this correspondence is bijective up to isomorphism. Now we ask if we also have a bijective correspondence between morphisms. The next proposition shows that this is the case.

If X and Y are Priestley spaces, their dual lattices are $\mathsf{ClpD}(X)$ and $\mathsf{ClpD}(Y)$. If we have an order-preserving continuous function $f : X \to Y$, how do we get a dual morphism between those dual lattices? We already saw the trick of swapping the direction of the arrow and considering the much better behaved preimage function instead of the direct image function: So given a clopen downset $B \subseteq Y$, we consider $A := f^{-1}(B) \subseteq X$. Since f is monotone, we already saw that this makes A again a downset; and since f is continuous, this makes A again clopen. Thus, we get a well-defined map $\mathsf{ClpD}(Y) \to \mathsf{ClpD}(Y)$. The next result shows that this is indeed a lattice homomorphism and, most importantly, that any every lattice homomorphism uniquely arises in this way—so we also have a bijective correspondence between morphisms.

Proposition 4.9. *Let* X *and* Y *be Priestley spaces. Let* $f : X \to Y$ *be an order-preserving continuous function. Then*

$$\mathsf{ClpD}(f) : \mathsf{ClpD}(Y) \to \mathsf{ClpD}(X)$$
$$B \mapsto f^{-1}(B)$$

is a lattice homomorphism. And if $h : \mathsf{ClpD}(Y) \to \mathsf{ClpD}(X)$ *is a lattice homomor-*

phism, there is a unique order-preserving continuous function $f : X \to Y$ *such that* $\mathsf{ClpD}(f) = h$.

This concludes the statement of the Priestley duality: as mentioned, it is the conjunction of propositions 4.3–4.9. Category-theorist would express them swiftly by saying:

Theorem 4.10 (Priestley duality). *The functors* Pr *and* ClpD *form a dual equivalence between, on the one side, the category* DL *of distributive lattices with lattice homomorphisms and, on the other side, the category* Priestley *of Priestley spaces with order-preserving continuous functions.*

4.2 Proof of the Priestley duality

In this section, we provide the proofs for the Priestley duality, i.e., propositions 4.3–4.9 above.

4.2.1 Constructing filters

Recall that prime filters F over a lattice L of properties are (recovered) objects. So, given a set of properties $A \subseteq L$, it would be very useful to construct prime filters F that contain all the properties in A (i.e., $A \subseteq F$). Because this means that we were able to construct an object F that has all the desired properties A. Surely this is not always possible, for example if A contains two inconsistent properties. But in this subsection we provide two results that show when this is possible. They correspond to two stages of the construction: the first result first extends A to a filter of L, and the second result then says when we can further extend this filter to a prime filter.

This is exercise 3.1.13 in Gehrke and van Gool 2023.

Proposition 4.11. *Let* L *be a lattice.*

1. *For any subset* $A \subseteq L$, *there is a* $\subseteq$*-smallest filter* F *that contains* A. *It is called the filter* generated by A *and denoted* $\langle A \rangle_{\text{filt}}$.

2. *Concretely, this filter is given as*

$$\langle A \rangle_{\text{filt}} = \big\{ a \in L : \text{ there is finite } A' \subseteq A \text{ such that } \bigwedge A' \leqslant a \big\}.$$

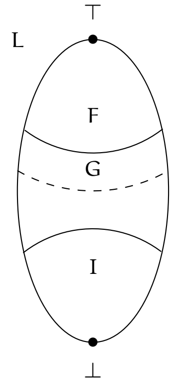
3. *If* $F \subseteq L$ *is a filter and we want to extend it by an element* $b \in L$, *this is concretely given as*

$$\langle F \cup \{b\} \rangle_{\text{filt}} = \big\{ a \in L : \text{ there is } f \in F \text{ such that } f \wedge b \leqslant a \big\}.$$

Order-dual results hold for ideals.

Theorem 4.12 (Stone's Prime Filter Extension Theorem)**.** *Let* L *be a distributive lattice. If* F *is a filter and* I *and ideal in* L *such that* $F \cap I = \emptyset$*, then there is a prime filter* G *in* L *such that* $F \subseteq G$ *and* $G \cap I = \emptyset$*.*

As a picture (cf. section 1.1.3):



The formulation using ideals makes this theorem more general: By choosing $I := \{\bot\}$, it says that any filter F not containing the inconsistent property $\bot$ can be extended to a prime filters. So the obstruction we mentioned before—that A is inconsistent—really is the only obstruction to extend A to a prime filter. But with the more general formulation in terms of ideal we can also, for example, take a property $b \notin \langle A \rangle_{\text{filt}}$ (i.e., b is not above a finite meet of properties in A) and, by choosing $I := \downarrow b$, construct a prime filter extension G that does not contain b.

Proof. We go for a typical Zorn's lemma argument: Order the potential candidates for a solution so that any maximal element is an actual solution. Here the candidates are the filters extending F that don't intersect I:

Zorn's lemma is equivalent to the axiom of choice and says: A (nonempty) partially ordered set containing upper bounds for every (nonempty) chain must have a maximal element.

$$\mathcal{P} := \left\{ G \in \mathsf{Filt}(L) : F \subseteq G \text{ and } G \cap I = \emptyset \right\}$$

which we order by inclusion. To apply Zorn's lemma, we need to check that $\mathcal{P}$ is nonempty—which it is, since $F \in \mathcal{P}$—and that any chain $\mathcal{C}$ in $\mathcal{P}$ has a maximal element: this is the case since $\bigcup_{G \in \mathcal{C}} G$ is a filter that belongs to $\mathcal{P}$ (verify this as an exercise). So Zorn's lemma applies and says that $\mathcal{P}$ has a maximal element G.

It remains to check that the maximality of G implies that it is prime. So assume $a \vee b \in G$ and show that either $a \in G$ or $b \in G$. The idea is to try and add a and b to G and conclude from the maximality of G that at least one of them must already be in G.

So consider the filter G_a (resp., G_b) generated by $G \cup \{a\}$ (resp. $G \cup \{b\}$). Recall from propositions 4.11 that it contains precisely those $c \in L$ for which there is $g \in G$ such that $g \wedge a \leqslant c$ (resp. $g \wedge b \leqslant c$). So G_a and G_b are still filter extending F and we claim that either $G_a \cap I = \emptyset$ or $G_b \cap I = \emptyset$. This implies $G \subseteq G_a, G_b \in \mathcal{P}$, so, since G is maximal, either $G = G_a$ or $G = G_b$, hence either $a \in G$ or $b \in G$, as needed.

Indeed, if there were $c \in G_a \cap I$ and $d \in G_b \cap I$, then $c \vee d$ is in I (since I is an ideal and hence closed under $\vee$) and both in G_a and in G_b (qua upsets). The latter implies that there is g_a and g_b in G such that $g_a \wedge a \leqslant c \vee d$ and $g_b \wedge b \leqslant c \vee d$. In particular, $g := g_a \wedge g_b \in G$ is such that $g \wedge a \leqslant c \vee d$

and $g \wedge b \leqslant c \vee d$. Hence, by distributivity,

$$g \wedge (a \vee b) = (g \wedge a) \vee (g \wedge b) \leqslant c \vee d.$$

Since $g \in G$ and $a \vee b \in G$, we have that the left-hand-side of the inequality is in G, and hence, qua upset, also the right-hand-side $c \vee d$ is in G. But this element also is in I, so $G \cap I \neq \emptyset$, contradiction. □

4.2.2 Proof of proposition 4.3

Let L be a distributive lattice. Write $X := \mathsf{PrFilt}(L)$ for the set of prime filters of L ordered by reverse inclusion (so $\leqslant$ is $\supseteq$) and equipped with the topology τ generated by all the $\widehat{a}$ and $\widehat{a}^c$ for $a \in L$. We need to show that $(X, \leqslant, \tau)$ is a Priestley space. For that we need to show compactness and the TOD property.

The TOD property is immediate: If $F \nleqslant G$ in X, then $F \nsupseteq G$, so there is $a \in G$ with $a \notin F$, so for the clopen downset $U := \widehat{a}$ of $\mathsf{PrFilt}(L)$, we have $G \in U$ and $F \notin U$.

So it remains to show compactness. We use the Alexander Subbase Theorem: Given an open cover $\mathcal{U}$ using the subbasic open sets $\widehat{a}$ and $\widehat{a}^c$, we need to find a finite subcover. So $\mathcal{U}$ is of the form $\{\widehat{a} : a \in A\} \cup \{\widehat{b}^c : b \in B\}$ for some $A, B \subseteq L$.

We first describe the intuitive idea of the proof before doing it formally. The key observation is that we cannot build an object G that has all the properties in B and none of the properties in A. Because if we could, then G must be in one set of the cover, so either $G \in \widehat{a}$ for some $a \in A$ or $G \in \widehat{b}^c$ for some property $b \in B$, but that means that the object G either has a property in A or does not have a property in B, which we excluded. This observation shows that the conjunction of properties in B implies some property a in A. As we expect in logic, already a finite subset B' of B hence should implies this property of A. But this means restricting the cover to $\{\widehat{a}\} \cup \{\widehat{b}^c : b \in B'\}$ yields a finite subcover: Any object which is not in any of the $\widehat{b}^c$ has all the properties in B' and hence has property a, i.e., is in $\widehat{a}$.

In logic, this is known as the compactness theorem *(if a set of premises implies a conclusion, already a finite subset of the premises does). Here, we get this from proposition 4.11. But it is no coincidence that logical compactness is related to spatial compactness.*

Now we do this formally. The trick of the proof—which now hopefully makes sense—is to consider the filter F generated by B and the ideal I generated by A, and to show that there must be a property $c \in F \cap I$ (which captures the idea that the properties in B imply some property in A): Indeed, if $F \cap I$ were empty, there is, by Stone's Prime Filter Extension Theorem (theorem 4.12), a prime filter G in L such that $F \subseteq G$ and $G \cap I = \emptyset$. Since $\mathcal{U}$ is a cover, either $G \in \widehat{a}$ for some $a \in A$ or $G \in \widehat{b}^c$ for some

$b \in B$. But both are impossible: The former cannot be, since otherwise $a \in A \subseteq I$ and $a \in G$, so $G \cap I \neq \emptyset$. And the latter cannot be, since otherwise $b \in B \subseteq F$ and $b \notin G$, so $F \nsubseteq G$.

Now for the compactness theorem idea: By proposition 4.11, since c is in the filter F generated by B, there is a finite $B' \subseteq B$ such that $\bigwedge B' \leqslant c$. Order-dually, for the ideal I generated by A, there is a finite $A' \subseteq A$ such that $c \leqslant \bigvee A'$. So $\bigwedge B' \leqslant \bigvee A'$.

This implies $\bigcap_{b \in B'} \widehat{b} \subseteq \bigcup_{a \in A'} \widehat{a}$: If F is a prime filter containing each element of B', then it also contains (by $\wedge$-closure) the element $\bigwedge B'$, and hence (by being an upset) also the element $\bigvee A'$, and hence (by being prime) some element of A', so F is in some $\widehat{a}$ with $a \in A'$.

So we can conclude as in the informal idea: This now means that for any prime filter F, if F is in no $\widehat{b}^c$ with $b \in B'$, then F is in each $\widehat{b}$ for $b \in B'$, and hence in some $\widehat{a}$ for $a \in A'$. So:

$$\mathsf{PrFilt}(L) = \bigcup_{a \in A'} \widehat{a} \cup \bigcup_{b \in B'} \widehat{b}^c.$$

Hence $\{\widehat{a} : a \in A'\} \cup \{\widehat{b}^c : b \in B'\}$ is a finite subcover of $\mathcal{U}$.

4.2.3 Proof of proposition 4.5

Let $(X, \tau, \leqslant)$ be a Priestley space. Let $L := \mathsf{ClpD}(X)$ be the set of clopen downsets of X ordered by inclusion. We have to show that L is a distributive lattice.

We show that L is a sublattice of the powerset lattice 2^X, which then also implies that it is distributive. Indeed, the empty set $\emptyset$ and the whole set X are clopen downsets. And if A and B are clopen downsets, also $A \cap B$ and $A \cup B$ are: this is because both open and closed sets are closed under finite intersection and finite union, and also downsets are closed under finite intersection and finite union.

4.2.4 Proof of proposition 4.7

Let L be a distributive lattice. We want to show that

$$\begin{aligned} \widehat{\cdot} : L &\to \mathsf{ClpD}(\mathsf{PrFilt}(L)) \\ a &\mapsto \widehat{a} = \{F \in \mathsf{PrFilt}(L) : a \in F\}. \end{aligned}$$

is a well-defined lattice isomorphism. We also write $X := \mathsf{PrFilt}(L)$.

The function is well-defined since $\widehat{a}$ is, by construction, a clopen downset of $\mathsf{PrFilt}(L)$. We first show that it is a lattice homomorphism:

It maps $\perp$ to $\widehat{bot} = \emptyset$ since prime filters are proper, and it maps $\top$ to $\widehat{\top} = X$ since prime filters are nonempty. Moreover, $\widehat{a \wedge b} = \widehat{a} \cap \widehat{b}$ since prime filters are closed under $\wedge$. And $\widehat{a \vee b} = \widehat{a} \cup \widehat{b}$ since prime filters are prime.

The function is injective: if $a \neq b$, we show $\widehat{a} \neq \widehat{b}$. By assumption, either $a \nleqslant b$ or $b \nleqslant a$. Without loss of generality, assume the former. Then $F := \uparrow a$ and $I := \downarrow b$ are a filter and ideal of L, respectively, with $F \cap I = \emptyset$. By Stone's Prime Filter Extension Theorem, there is a prime filter G in L such that $F \subseteq G$ and $G \cap I = \emptyset$. So $a \in F \subseteq G$ and $b \notin G$ (otherwise $b \in G \cap I$). So $G \in \widehat{a}$ but $G \notin \widehat{b}$, as needed.

Finally, we show that the function is surjective, for which we make use of the compactness of X. Let $A \subseteq X$ be a clopen downset and find $a \in L$ with $\widehat{a} = A$. For every pair (F, G) of points in $X = \mathsf{PrFilt}(L)$, there is, as we've seen in the proof of the TOD property of X, some $a_{(F,G)} \in L$ with $G \in \widehat{a_{(F,G)}}$ and $F \notin \widehat{a_{(F,G)}}$. We now use a compactness argument twice (once for F and once for G, so to speak).

First, for every $F \in A$, we have the following open cover of A^c:

$$\left\{\widehat{a_{(F,G)}}^{\,c} : G \in A^c\right\}$$

Since A^c is a closed subset of the compact space X, it is compact, so there is a finite subcover $\{\widehat{a_{(F,G_1)}}^{\,c}, \dots, \widehat{a_{(F,G_n)}}^{\,c}\}$. Define $a_F := \bigwedge i = 1^n a_{(F,G_i)}$. Then $F \in \widehat{a_F}$ because, since $\widehat{\cdot}$ is a lattice homomorphism, we have $\widehat{a_F} = \bigcap_{i=1}^n \widehat{a_{(F,G_i)}} \in L$, and for any i, since $F \notin A^c$, F is not in $\widehat{a_{(F,G_i)}}^{\,c}$, so it is in $\widehat{a_{(F,G_i)}}$. Moreover, $\widehat{a_F} \subseteq A$: if $G \in \widehat{a_F}$ but $G \in A^c$, then G is in some $\widehat{a_{(F,G_i)}}^{\,c}$, so $a_F \in G$ but $a_{(F,G_i)} \not inG$, despite $a_F \leqslant a_{(F,G_i)}$, which contradicts G being an upset.

Second, now we have, for every $F \in A$, the open set $\widehat{a_F}$ which contains F and is a subset of A. So $\{\widehat{a_F} : F \in A\}$ is an open cover of A. Since A is a closed subset of the compact space X, it is compact, so there is a finite subcover $\{\widehat{a_{F_1}}, \dots, \widehat{a_{F_m}}\}$. Since all sets of the cover are subsets of A, their union is A: $A = \bigcup_{j=1}^m \widehat{a_{F_j}}$.

Now, set $a := \bigvee_{j=1} a_{F_j} \in L$. Then, since $\widehat{\cdot}$ is a lattice homomorphism, $A = \bigcup_{j=1}^m \widehat{a_{F_j}} = \widehat{a}$, as needed.

4.2.5 Proof of proposition 4.8

Let $(X, \tau, \leqslant)$ be a Priestley space. We want to show that

$$\begin{aligned}\beta : X &\to \mathsf{PrFilt}(\mathsf{ClpD}(X))\\ x &\mapsto \{a \in \mathsf{ClpD}(X) : x \in a\}\\ \text{the greatest } x \text{ with } x \in a \text{ for all } a \in F &\mapsfrom F\end{aligned}$$

is a well-defined order homeomorphism. To do so, we first prove two independently interesting lemmas. Also recall the TOD property, which we'll heavily use: If $x \not\leqslant y$, there is a clopen downset U in X such that $y \in U$ and $x \notin U$.

Lemma 4.13. *For any prime filter $F \subseteq \mathsf{ClpD}(X)$, there is $x \in X$ such that $\bigcap F = {\downarrow} x$.*

Proof. To show that a 'big intersection' is nonempty, the classic trick is to use the 'finite intersection property' characterization of compactness. Indeed, consider the following collection of closed (in fact, clopen) sets:

$$\mathcal{C} := \big\{a \in \mathsf{ClpD}(X) : a \in F\} \cup \big\{a^c : a \in \mathsf{ClpD}(X) \setminus F\}.$$

We will soon see why we not just include the sets of F but the complements of F^c.

We show that $\mathcal{C}$ has the finite intersection property. Indeed, if $\mathcal{C}'$ is a finite subset of $\mathcal{C}$, it is of the form $\{a_1, \ldots, a_n, b_1^c, \ldots, b_m^c\}$ for $a_1, \ldots, a_n \in F$ and $b_1, \ldots, b_m \in F^c$. If this had an empty intersection, we would have

$$a_1 \cap \ldots \cap a_n \subseteq (b_1^c \cap \ldots \cap b_m^c)^c = b_1 \cup \ldots \cup b_m,$$

but then the left-hand-side is in F, so, qua upset, also the right-hand-side is in F, hence, qua prime filter, some b_j is in F, but $b_j \in F^c$, contradiction.

Now, since X is compact, it follows (see the end of section 3.1.1) that $\bigcap \mathcal{C}$ is nonempty, so let $x \in \bigcap \mathcal{C}$. It remains to show that $\bigcap F = {\downarrow} x$.

($\supseteq$) Since $F \subseteq \mathcal{C}$, we have $x \in \bigcap F$, and since F is a downset (qua intersection of downsets) also ${\downarrow} x \subseteq \bigcap F$.

($\subseteq$) Let $y \in \bigcap F$ and show $y \leqslant x$. We use the contrapositive of the TOD property: so we have to show that for all $a \in \mathsf{ClpD}(X)$, if $x \in a$, then $y \in a$. Indeed, if $a \in \mathsf{ClpD}(X)$ with $x \in a$, then $a \in F$, because otherwise $a \in F^c$, so, since $x \in \bigcap \mathcal{C}$, we would have $x \in a^c$. Since $y \in \bigcap F \subseteq a$, we have $y \in a$. ∎

Lemma 4.14. *Let* $(X, \tau, \leqslant)$ *be a Priestley space. Then*

$$\mathcal{B} = \{A \setminus B : A, B \in \mathsf{ClpD}(X)\}$$

is a base for τ.

Proof. Qua finite intersection of clopens, each $A \setminus B$ is open. So $\mathcal{B}$ is a collection of opens, and to show that it is a base, take an open set U and a point $x \in U$, and find an element of $\mathcal{B}$ that contains x and is a subset of U.

Now we make a compactness argument. Note that, for each $y \in U^c$, either $x \nleqslant y$ or $x \ngeqslant y$ (and not both). By the TOD property, either there is a clopen downset B_y containing y but not x, or there is a clopen downset A_y containing x but not y (so A_y^c contains y but not x). So

$$\left\{A_y^c : y \in U^c, x \ngeqslant y\right\} \cup \left\{B_y : y \in U^c, x \nleqslant y\right\}$$

is an open cover of U^c. Since U_c is a closed subset of the compact space X, it is compact, so there is a finite subcover

$$\{A_{y_1}^c, \ldots, A_{y_n}^c\} \cup \{B_{y_n+1}, \ldots, B_{y_m}\}. \tag{4.1}$$

Define $A := \bigcap_{i=1}^{n} A_{y_i}$ and $B := \bigcup_{j=n+1}^{m} B_{y_j}$. Then

- A and B are clopen downsets of X (qua finite intersections and unions of such sets),
- $x \in A \cap B^c$, since x is in every A_{y_i} and in no B_{y_j},
- $A \cap B^c \subseteq U$, because, for $z \in X$, if $z \notin U$, then, since (4.1) is a cover, z either is in some $A_{y_i}^c$ or in some B_{y_j}. But then, in the former case z is not in A and in the latter case z is not in B^c, so $z \notin A \cap B^c$.

Hence $A \setminus B \in \mathcal{B}$ and $x \in A \setminus B \subseteq U$, as needed. □

Now we show that β is an order homeomorphism. To show β is well-defined, note that $\{a \in \mathsf{ClpD}(X) : x \in a\}$ is indeed a prime filter of $\mathsf{ClpD}(X)$: it is a nonempty upset closed under intersection, doesn't contain $\emptyset$ and if $x \in a \cup b$, then either $x \in a$ or $x \in b$.

So we need to show that β is (1) continuous, (2) order-preserving, (3) order-respecting, (4) surjective, and (5) open. We also write $L := \mathsf{ClpD}(X)$. For that, we first observe that, by construction,

$$\forall a \in L \forall x \in X : x \in a \Leftrightarrow a \in \beta(x) \Leftrightarrow \beta(x) \in \widehat{a} \tag{4.2}$$

Ad (1). For any subbasic open sets $\widehat{a}$ of $\mathsf{PrFilt}(L)$, we have, by (4.2), $a = \beta^{-1}(\widehat{a})$. So preimages of subbasic opens are open, hence β is continuous.

Ad (2). If $x \leqslant y$, we show $\beta(x) \leqslant \beta(y)$, i.e., $\beta(x) \supseteq \beta(y)$. If $a \in \beta(y)$, then a is a clopen downset of X with $y \in a$; since $x \leqslant y$ and a is a downset, also $x \in a$, so $a \in \beta(x)$, as needed.

Ad (3). If $x \nleqslant y$, we need to show $\beta(x) \nleqslant \beta(y)$. This is precisely the TOD property of X: if $x \nleqslant y$, there is a clopen downset $a \in \mathsf{ClpD}(X)$ such that $y \in a$ and $x \notin a$, so $\beta(x) \not\supseteq \beta(y)$.

Ad (4). Let $F \in \mathsf{PrFilt}(\mathsf{ClpD}(X))$, then, by the lemma, there is $x \in X$ such $\bigcap F = {\downarrow}x$. We show that $\beta(x) = F$. If $a \in F$, then $x \in \bigcap F \subseteq a$, so, by (4.2), $a \in \beta(x)$. For the other direction, we make a compactness argument. Assume $a \in \beta(x)$ and show $a \in F$. By the assumption, $x \in A$, so, since $\bigcap F = {\downarrow}x$ and a is a downset, also $\bigcap F \subseteq a$. Now $\{f^c : f \in F\}$ is an open cover of the closed—and hence compact—subset $a^c \subseteq X$ (if $y \in a^c$, then $y \in (\bigcap F)^c = \bigcup_{f \in F} f^c$). So there are finitely many $f_1, \ldots, f_n \in F$ such that $f_1^c \cup \ldots \cup f_n^c \supseteq a^c$, so

$$f_1 \cap \ldots \cap f_n = \left(f_1^c \cup \ldots \cup f_n^c\right)^c \subseteq a.$$

Since the left-hand-side is in F, and F is an upset, also $a \in F$, as needed.

Ad (5). If $U \subseteq X$ is open, we need to show that $\beta[U]$ is open. First, if $U = a$ is a clopen downset of X, then $\beta[a] = \widehat{a}$, so the image of U is (cl)open: Indeed, given $\beta(x)$ for $x \in a$, we have, by (4.2), $\beta(x) \in \widehat{a}$. If $F \in \widehat{a}$, then, by bijectivity, $F = \beta(x)$ for the greatest element x in $\bigcap F$; and since $a \in F$ (because $F \in \widehat{a}$), we hence have $x \in \bigcap F \subseteq a$, so $F = \beta(x) \in \beta[a]$.

Second, if $U = a \setminus b$ for clopen downsets a and b of X, then, since β is bijective, $\beta[U] = \beta[a] \setminus \beta[b]$, hence open qua finite intersection of two clopen sets.

Third, this now extends to all open sets U: By lemma 4.14 U is a union of sets of the form $a \setminus b$, so the image of U is the union of the images of these sets, which, as just seen, are open, hence also their union is open.

4.2.6 Proof of proposition 4.9

Let $f : X \to Y$ be an order-preserving continuous function between Priestley spaces. We need to show (1) that

$$\begin{aligned} \mathsf{ClpD}(f) : \mathsf{ClpD}(Y) &\to \mathsf{ClpD}(X) \\ B &\mapsto f^{-1}(B) \end{aligned}$$

is a lattice homomorphism. And (2) if $h : \mathsf{ClpD}(Y) \to \mathsf{ClpD}(X)$ is a lattice homomorphism, there is a unique order-preserving continuous function $f : X \to Y$ such that $\mathsf{ClpD}(f) = h$.

Concerning (1), we have already argued in section 4.1.5 that this is well-defined (i.e., $f^{-1}(B)$ is again a clopen downset). And we already observed the good preservation properties of the preimage map: it preserves intersections and unions, so this also is a lattice homomorphism.

So it remains to show (2). The uniqueness claim is easy: Assume $f, f' : X \to Y$ are order-preserving and continuous with $\mathsf{ClpD}(f) = h = \mathsf{ClpD}(f')$, and show $f = f'$. If not, there is $x \in X$ with $y := f(x) \neq f'(x) =: y'$. So either $y \not\leq y'$ or $y' \not\leq y$. Without loss of generality, assume the former case. Then the TOD property implies that there is $B \in \mathsf{ClpD}(Y)$ such that $y' \in B$ but $y \notin B$. So $x \in f'^{-1}(B)$ but $x \notin f^{-1}(B)$. Hence $\mathsf{ClpD}(f)(B) = f^{-1}(B) \neq f'^{-1}(B) = \mathsf{ClpD}(f')(B)$, contradiction.

For the existence claim, we need to find a function $f : X \to Y$ such that $\mathsf{ClpD}(f) = h$. The idea is to use the double duals of X and Y via the isomorphisms $\beta_X : X \to \mathsf{PrFilt}(ClpD(X))$ and $\beta_Y : Y \to \mathsf{PrFilt}(ClpD(Y))$. Recall that if $x \in X$, then $\beta_X(x) = \{a \in \mathsf{ClpD}(X) : x \in a\}$ is a prime filter of $\mathsf{ClpD}(X)$. The idea is to consider $h^{-1}(\beta_X(x)) \subseteq \mathsf{ClpD}(Y)$ and hope that this again is a prime filter of $\mathsf{ClpD}(Y)$, so under β_Y, it corresponds to a unique point y of Y. Then we choose $f(x) := y$. We now check that this idea works.

First, we get that $h^{-1}(\beta_X(x))$ is indeed a prime filter by the following general lemma, whose proof is a good exercise.

Lemma 4.15. *Let* $h : L \to M$ *be a lattice homomorphism and* $F \subseteq M$ *a prime filter. Then* $h^{-1}(F) \subseteq L$ *is a prime filter.*

Prove this as an exercise.

So we can define the function

$$\begin{aligned} f : X &\to Y \\ x &\mapsto \beta_Y^{-1}\big(h^{-1}(\beta_X(x))\big) \end{aligned}$$

And it remains to check that (a) f is continuous, (b) f is order-preserving, and (c) $\mathsf{ClpD}(f) = h$. To do so, we first note that, for $B \in \mathsf{ClpD}(Y)$ and $x \in X$, we have

$$x \in f^{-1}(B) \Leftrightarrow x \in h(B). \tag{4.3}$$

Indeed, first note that $x \in f^{-1}(B)$ is equivalent to $f(x) \in B$, which in turn is equivalent, since β_Y is bijective, to $\beta_Y(f(x)) \in \beta_Y[B]$. Now, since β_Y

is bijective, we have $\beta_Y(f(x)) = h^{-1}(\beta_X(x))$. And from the proof that β_Y is an isomorphism (section 4.2.5) we know that $\beta_Y[B] = \widehat{B} = \{G \in \mathsf{PrFilt}(\mathsf{ClpD}(Y)) : B \in G\}$. Hence, so far, we have

$$x \in f^{-1}(B) \Leftrightarrow h^{-1}(\beta_X(x)) \in \widehat{B}.$$

And by the respective definitions, we have the further equivalences

$$\begin{aligned} h^{-1}(\beta_X(x)) \in \widehat{B} &\Leftrightarrow B \in h^{-1}(\beta_X(x)) \\ &\Leftrightarrow h(B) \in \beta_X(x) = \{A \in \mathsf{ClpD}(X) : x \in A\} \Leftrightarrow x \in h(B), \end{aligned}$$

as needed.

Ad (a). By lemma 4.14, we know that, for $A, B \in \mathsf{ClpD}(Y)$, the set $A \setminus B$ is basic open. So we need to show that $f^{-1}(A \setminus B) \subseteq X$ is open. Indeed, by (4.3), we have $f^{-1}(A \setminus B) = f^{-1}(A) \cap f^{-1}(B)^c = h(A) \cap h(B)^c$. Since $h(A)$ and $h(B)$ are clopen, this is a clopen and hence open set.

Ad (b). If $x \leqslant x'$ in X, then, because β_X is order-preserving, $\beta_X(x) \leqslant \beta_X(x')$, i.e., $\beta_X(x) \supseteq \beta_X(x')$, so $h^{-1}(\beta_X(x)) \supseteq h^{-1}(\beta_X(x'))$, so, since β_Y is order-reflecting, $f(x) = \beta_Y^{-1}\big(h^{-1}(\beta_X(x)\big) \leqslant \beta_Y^{-1}\big(h^{-1}(\beta_X(x')\big) = f(x')$.

Ad (c). For $B \in \mathsf{ClpD}(Y)$, we have, by (4.3), that $\mathsf{ClpD}(f)(B) = f^{-1}(B) = h(B)$.

4.3 Stone duality

Having seen Priestley duality, we now move to the variant of it due to Stone. It relates every distributive lattice to a spectral space (definition 3.11) instead of a Priestley space (definition 3.10). So it does not make use of the order relation, but the price of this simplification is that the topology is more complicated. We restrict us here to stating the duality, without proving it.

4.3.1 The Stone duality ...

The Stone duality translates between lattices and spaces as follows:

- If L is a distributive lattice, the dual spectral space (X, τ) is given by X the set of prime filters of L and τ generated by the sets $\widehat{a} = \{F \in X : a \in F\}$. This space is denoted $\mathsf{St}(L)$ and is indeed a spectral space.

  If $h : L \to M$ is a lattice homomorphism, then $\mathsf{St}(f) : \mathsf{St}(M) \to \mathsf{St}(L)$ maps a prime filter G of M to the prime filter $h^{-1}(G) = \{a \in L :$

So this is just like the dual Priestley space except that we don't add the complements $\widehat{a}^c$ as open sets.

$h(a) \in G\}$. This is indeed a spectral map (continuous and preimages of compact opens are compact).

- If (X, τ) is a spectral space, the dual distributive lattice is $\mathsf{KO}(X)$, the set of compact-open subsets of X ordered by inclusion.

  If $f : X \to Y$ is a spectral map between spectral spaces, then $\mathsf{KO}(f) : \mathsf{KO}(Y) \to \mathsf{KO}(X)$ maps a compact-open set B of Y to the preimage $f^{-1}(B)$. This is a lattice homomorphism.

Now one can show that these again form a duality: every distributive lattice L is isomorphic to its double-dual $\mathsf{KO}(\mathsf{St}(L))$ and every spectral space X is homeomorphic to its double dual $\mathsf{St}(\mathsf{KO}(X))$; and for all lattice homomorphisms $h : \mathsf{KO}(Y) \to \mathsf{KO}(X)$ there is a unique spectral map $f : X \to Y$ such that $h = \mathsf{KO}(f)$. Again, formulated in category-theoretic language this means:

Theorem 4.16 (Stone duality)**.** *The functors* St *and* KO *form a dual equivalence between, on the one side, the category* DL *of distributive lattices with lattice homomorphisms and, on the other side, the category* Spectral *of spectral spaces with spectral maps.*

4.3.2 ... and its relation to Priestley duality

And here is how this Stone duality is very closely related to the Priestley duality: the categories of spaces that they use are in fact isomorphic.

- If $(X, \tau, \leqslant)$ is a Priestley space, define

  $$\tau^{\downarrow} := \{U \in \tau : U \text{ is a } \leqslant\text{-downset}\}.$$

  So $\tau^{\downarrow}$ is the meet of the topology τ and the dual Alexandrov topology.

- If τ is a topology on a set X, the *co-compact dual* of τ is generated by the complements of compact-saturated subsets:

  $$\tau^{\delta} := \langle \{K^c : K \in \mathsf{KS}(X, \tau)\} \rangle$$

  If X is a stably compact space (in particular, a spectral space), $\mathsf{KS}(X)$ is not only closed under finite union but also under arbitrary intersection, so the generating set already is a topology. Finally, the *patch topology* is defined as

  $$\tau^{p} := \tau \vee \tau^{\delta}.$$

- The assignments

$$\begin{aligned} \mathsf{Priestley} &\leftrightarrows \mathsf{Spectral} \\ (X, \tau, \leqslant) &\mapsto (X, \tau^{\downarrow}) \\ (X, \rho^{p}, \geqslant_{\rho}) &\mapsfrom (X, \rho) \end{aligned}$$

  form a bijective correspondence between Priestley spaces and spectral spaces. In fact, this is an isomorphism on categories which, on morphisms, simply is the identity.

See Gehrke and van Gool (2023, thm. 6.4).

Recall that $\leqslant_\rho$ is the specialization order of the topology ρ; and for Priestley spaces we were working with its dual $\geqslant_\rho$, the generalization order.

And all these constructions commute: For example, if we start with a distributive lattice L and build the dual Priestley space $(X, \tau, \leqslant) := \mathsf{Pr}(L)$ and the the corresponding spectral space $(X, \tau^{\downarrow})$ this is the same as if we had built the dual spectral space $\mathsf{St}(L)$ directly. Or if we start with a spectral space (X, τ), build the corresponding Priestley space $(X, \tau^{p}, \geqslant_{\tau})$ and then the dual lattice, it is the same as building the dual lattice $\mathsf{KO}(X, \tau)$ directly. And similarly for other combinations. Formally, this is expressed by saying that the top triangle of figure 4.3 commutes.

4.4 The Boolean case

In this section, we see how the Priestley/Stone duality restricts when considering Boolean algebras. The corresponding spaces are known as Stone spaces (or also Boolean spaces). This duality may be viewed as literally the Stone duality restricted to Boolean algebras or—as we will see—as Priestley duality 'without the order'.

4.4.1 From Boolean algebras to Stone spaces

If we start the duality with a distributive lattice L that in fact is a Boolean algebra, what does the dual Priestley space $\mathsf{Pr}(L)$ look like? Just like in the finite case (section 2.4.4), the key insight is that then the order on the dual space is trivial: if F and G are two primes filter with $F \subseteq G$, then already $F = G$. This is implied by the following proposition, which in fact gives a well-known characterization of prime filters in Boolean algebras. The proof is a recommended exercise (exercise 4.b).

Proposition 4.17. *Let A be a Boolean algebra, and let $F \subseteq A$ be a filter. Then the following are equivalent.*

1. *F is a prime filter.*

2. F *is a* maximal *filter, i.e.,* F *is proper and for any proper filter* G *with* $F \subseteq G$, *we have* $F = G$.

3. F *is an* ultrafilter, *i.e.,* F *is proper and for any* $a \in A$, *either* $a \in F$ *or* $\neg a \in F$.

The philosophical meaning of this is that the usual, well-motivated notion of a prime filter is equivalent to other common notions of a 'model' in classical logic (and Boolean algebras are the algebraic version of classical logic). Prime filters require the models to respect conjunction and disjunction. Ultrafilters require models to respect conjunction and negation. Maximal filters require models to be maximally consistent. All three are common ways of specifying a classical model or object or possible world (or whatever the philosophical interpretation of the points in the space).

Cf. section 2.3.2: being a lattice homomorphism (respecting $\wedge$ and $\vee$) already implies being a Boolean algebra homomorphism (i.e., also respecting $\neg$).

So if we start with a Boolean algebra L, the order of the dual Priestley space $\mathsf{Pr}(L)$ is trivial. So it makes sense to give those Priestley spaces a name. They are known as *Stone spaces* or as *Boolean spaces*. The former name might be more common, but 'Boolean space' is clearer since some also refer to spectral spaces as Stone spaces.

Definition 4.18. A *Stone* (aka *Boolean*) *space* is a topological space (X, τ) that is compact and *totally disconnected*, i.e., for any $x \neq y$ in X, there is a clopen set $U \subseteq X$ such that $x \in U$ and $y \notin U$.

Verify for yourself that this is what remains of the TOD property when the order is trivial.

An equivalent characterization of Stone spaces (see exercise 4.c) is as topological spaces which are compact, Hausdorff, and *zero-dimensional*; here zero-dimensional means that the clopens form a base. This also is often used as a definition.

But—one might wonder—what if we had used Stone duality to move from the Boolean algebra L to the dual space? Do we then also get a Stone space? Given the equivalence of Priestley spaces and spectral spaces, one would hope so, but let's double check: The dual space $\mathsf{St}(L)$ is given by the set X of prime filters of L with the topology generated by $\widehat{a}$. For general distributive lattice, the complement need not be open, but for Boolean algebras it is: For $a \in L$, we have, using the ultrafilter characterization of prime filters,

$$\widehat{a}^c = \{F \in X : a \notin F\} = \{F \in X : \neg a \in F\} = \widehat{\neg a}.$$

So (X, τ) is compact qua spectral space but it also is totally disconnected: if $F \neq G$, then there is $a \in F$ with $a \notin G$ (or vice versa), so $\widehat{a} \subseteq X$ is a clopen set such that $F \in \widehat{a}$ and $G \notin \widehat{a}$.

4.4.2 From Stone spaces to Boolean algebras

If we have a Stone space X and think of it as a Priestley space with a trivial order, its dual lattice is the set of all clopen downsets ordered by inclusion. But, since the order is trivial, these simply are the downsets. They indeed form a Boolean algebra, because the negation is given by the set-theoretic complement.

If we think of X as a spectral space, its dual lattice is the set of all its compact-opens. But since X is a compact Hausdorff space, compact subsets are closed, and closed subset are compact (see exercise 3.f). So the compact-opens coincide with the clopens. So the dual lattice again just is the set of clopens ordered by inclusion. To summarize:

Proposition 4.19. *If X is a Stone space, then $\mathsf{KO}(X) = \mathsf{Clp}(X)$ is a Boolean algebra.*

Finally, what is the appropriate notion of morphism for Stone spaces? Coming from Priestley spaces, they should be order-preserving continuous maps, but since the order is trivial, this reduces simply to continuous maps. Coming from spectral spaces, the morphisms should be spectral maps, but since preimages of compact-open sets are automatically compact (since the compact-opens coincide with the clopens), the specttral maps also reduce to simply continuous maps. So continuous maps are the appropriate morphisms of Stone spaces.

We again can summarize the discussion with the following duality.

Theorem 4.20 (Stone duality for Boolean algebras)**.** *The functors St and Clp form a dual equivalence between, on the one side, the category BA of Boolean algebras with Boolean algebras homomorphisms and, on the other side, the category Stone of Stone spaces with continuous maps.*

Figure 4.3 summarizes all the duality results we have covered in this chapter.

4.5 Exercises

Exercise 4.a. Let L be a lattice and $F \subseteq L$. Show that the following are equivalent.

1. The set F is a prime filter.
2. The complement $I := L \setminus F$ is a prime ideal.
3. The characteristic function $\chi_F : L \to \mathbf{2}$ is a lattice homomorphism.

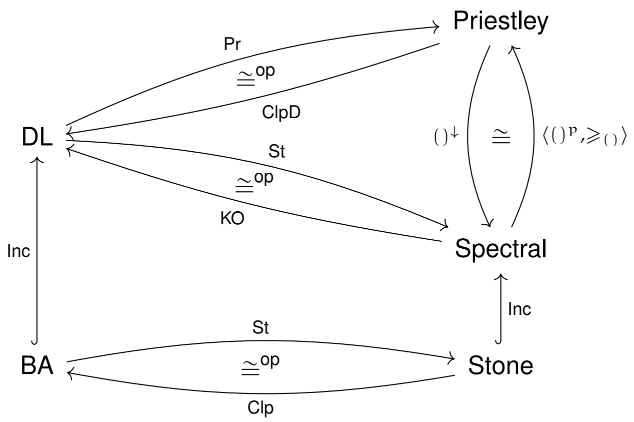


Figure 4.3: The Stone and Priestley duality formally

Exercise 4.b. Prove the characterization of prime filters in Boolean algebras stated in proposition 4.17.

Exercise 4.c. Let (X, τ) be a topological space. Prove that the following are equivalent.

1. (X, τ) is a Stone space in the sense of definition 4.18, i.e., compact and totally disconnected.

2. (X, τ) is compact, Hausdroff, and zero-dimensional.

5 Applications

There are a plethora of applications of the duality results: just take a look at chapters 4–8 of Gehrke and van Gool (2023). Here we will pick out one: modal logic. We choose this because it showcases an application that is particular useful in a philosophical context. We add a less understood logical concept—here, necessity—to our logical concepts, and we get a better understanding of it by studying it on the dual side of spaces.

5.1 (Re)discovering the semantics of 'necessity'

So assume we have our nice set L of propositions with the usual Boolean connectives $\wedge, \vee, \neg$. Thus, L is a Boolean algebra. Its elements are the meanings of declarative sentences. The connectives represent the special logical role played by conjunction, disjunction, and negation of sentences: For example, if sentence φ expresses proposition a and the sentences ψ expresses proposition b, then the sentence 'φ and ψ' expresses the proposition $a \wedge b$. Since the two sentences 'φ and ψ' and 'ψ and φ' have the same meaning, even if they are syntactically distinct, we have $a \wedge b = b \wedge a$, and similarly for the other laws of a Boolean algebra.

Analogously for $\vee$ and $\neg$.

As philosophers, we're not quite happy yet with this choice of logical connectives since it is not yet expressive enough. For philosophical discussions, it is also important whether a sentence is necessarily or just accidentally true. If if φ is a sentence expressing proposition a, we also consider the sentence 'Necessarily, φ'—customarily written as $\Box\varphi$—which expresses the proposition that it is necessarily the case that a. So just like the Boolean logical connectives translate to functions on the set L of propositions, also our new necessity connective translates to a function

$$\Box : L \to L.$$

As with negation, we typically write $\Box a$ instead of $\Box(a)$.

This already tells us a little bit about 'necessity': namely, the type of thing it is (a function that maps propositions to propositions). But we don't know yet what it really means: we don't know its identity.This is the situation that philosophers also were in before Kripke and others (in the late 1950s).

5.1.1 The search for truth-conditions of 'necessity'

You might already know the Kripke semantics for modal logic, which is (among others) the logic of the necessity connective. If you do, then forget it again for now—and if you don't, even better—because we want to rediscover it using the tools of duality theory. Before the advent of Kripke semantics, philosophers—like us now—struggled to understand the necessity connective. For the Boolean connective, they could say what their meaning is: for example, they could point to their truth-tables. In more fancy terminology, they had compositional truth-conditions for the Boolean connective: the sentence $\varphi \vee \psi$ is true (in some situation) iff either φ is true (in that situation) or ψ is true (in that situation). However, for 'necessity', they could only provide some plausible reasoning principles, like the following.

1. It is necessary that φ and ψ ($\Box(\varphi \wedge \psi)$) if and only if it is necessary that φ and it is necessary that ψ ($\Box\varphi \wedge \Box\psi$).

2. If φ is a logical truth, then 'it is necessary that φ' also is a logical truth.

So the situation was as if philosophers could point to reasoning rules for disjunction like 'If φ, then $\varphi \vee \psi$', but not to the truth-table or truth-conditions of $\vee$.

Now, how can duality theory help to get such an understanding of the meaning of 'necessity'? The trick is—as always when it comes to applications of duality theory—to move to the dual side and hope for clearer intuitions there. So let's do this.

All we know so far is that $\Box$ is some (so far unknown) function from L to L which—by stating the principles 1 and 2 above more formally—satisfies:

1. For all $a, b \in L$: $\Box(a \wedge b) = \Box a \wedge \Box b$.

2. $\Box\top = \top$.

How can we translate this function $\Box : L \to L$ to the dual side involving the Stone space $\mathsf{St}(L)$ of ultrafilters on L? (We described this in sections 4.3 and 4.4.) We already know that we can think of these ultrafilters as (Ersatz) possible worlds: they are maximally consistent sets of sentences. And we know that a function $h : L \to M$ on Boolean algebras that in fact is a lattice homomorphism translates to a continuous function $\mathsf{St}(h) : \mathsf{St}(M) \to \mathsf{St}(L)$

by mapping the ultrafilter G on M to the ultrafilter $h^{-1}(G) = \{a \in L : h(a) \in G\}$. Unfortunately, our $\Box$ is something more general than a lattice homomorphism: properties 1 and 2 only state preservation of $\top$ and $\wedge$, but not of $\bot$ and $\vee$. But maybe we can still translate it to something more general than a continuous function on the dual space? The more general concept than a function is a relation. And if we can translate $\Box$ to a relation on the dual space, then we have an interpretation of $\Box$ as something that relates possible worlds—and that seems promising to get the compositional truth-conditions that we are aiming to find.

Let's see how far we can push the idea of Stone duality of translating a lattice homomorphism $h : L \to L$ to the function $\mathsf{St}(h)$ which relates an ultrafilter F of L to the ultrafilter $h^{-1}(F)$ of L. Thinking of $\mathsf{St}(h)$ as a relation, we have, for any ultrafilters $F, G \in \mathsf{St}(L)$, the equivalences:

	$(F, G) \in \mathsf{St}(h)$	
iff	$G = h^{-1}(F)$	definition of the function $\mathsf{St}(h)$
iff	$G \supseteq h^{-1}(F)$	ultrafilters are maximal
iff	$\forall a \in L : h(a) \in F \Rightarrow a \in G$	by definition of the preimage.

Admittedly, at this point it's not clear why to move to $\supseteq$ and not $\subseteq$ or keep $=$. But this will make theorem 5.1 below work. In fact, it's the way to get this (Gehrke and van Gool 2023, prop. 4.39). So at least in hindsight, this is the only choice.

So for our function $\Box : L \to L$, let's consider the dual relation

$$F \mathrel{R} G :\text{ iff } \forall a \in L : \Box a \in F \Rightarrow a \in G. \tag{5.1}$$

Then what does it mean to say that $\Box a$ is true at the possible world F, i.e., $\Box a \in F$? In other words, what are the truth-conditions for $\Box a$? The following provides the answer.

That's the punchline of this section. Do you recognize this? (To be revealed below.)

Theorem 5.1. *In the preceding notation, we have for all $a \in L$ and ultrafilters $F \in \mathsf{St}(L)$*

$$\Box a \in F \Leftrightarrow \forall G \in \mathsf{St}(L) : F \mathrel{R} G \Rightarrow a \in G.$$

Before proving this, note that if you have seen Kripke semantics, this is *exactly* the truth-condition for the necessity connective:

- $\Box a$ is true at a possible world F iff for all R-accessible worlds G, we have that a is true at G.

So we have a way of relating the truth of the complex proposition $\Box a$ to the truth of its constituent a: hence this truth-condition is compositional. Recall, by the properties of ultrafilters, we also have compositional truth-conditions for the Boolean connectives.

- $a \wedge b$ is true at a possible world F (i.e., $a \wedge b \in F$) iff both a is true at F (i.e., $a \in F$) and b is true at F (i.e., $b \in F$).
- $a \vee b$ is true at a possible world F (i.e., $a \vee b \in F$) iff either a is true at F (i.e., $a \in F$) or b is true at F (i.e., $b \in F$).
- $\neg a$ is true at a possible world F (i.e., $\neg a \in F$) iff a is not true at F (i.e., $a \notin F$).

So we have truth-conditions for all the connectives. Now for the proof.

Proof. ($\Rightarrow$) If $\Box a \in F$ and $F \mathrel{R} G$, then, by definition 5.1, we have $a \in G$.

($\Leftarrow$) Contrapositively, assume $\Box a \notin F$. We want to find an ultrafilter $G \in \mathsf{St}(L)$ with $F \mathrel{R} G$ but $a \notin G$. So of course we turn to the Prime Filter Extension Theorem (theorem 4.12). Consider

$$G_0 := \{b \in L : \Box b \in F\}$$

We check that this is a filter that does not contain a. Upset: If $b \in G_0$ and $b \leqslant c$, then $\Box b \in F$ and $b \wedge c = b$, so, by 1, $F \ni \Box b = \Box(b \wedge c) = \Box b \wedge \Box c$, hence, since F is an upset, $\Box c \in F$, so $c \in G_0$. Closure under conjunction: If b and c are in G_0, then $\Box b$ and $\Box c$ are in F, so $\Box a \wedge \Box b$ is in F, which is, by 1, identical to $\Box(a \wedge b)$, so $a \wedge b$ is in G_0. Nonempty: By 2, $\Box\top = \top \in F$, so $\top \in G_0$. Doesn't contain a: because by assumption $\Box a \notin F$.

Now we can apply the Prime Filter Extension Theorem to get an ultrafilter G extending G_0 that still does not contain a. So it remains to check $F \mathrel{R} G$. Indeed, for $b \in L$, if $\Box b \in F$, then $b \in G_0 \subseteq G$. $\Box$

5.1.2 Kripke semantics

Now, with that hindsight, the usual Kripke semantics for modal logic almost seems obvious: We use the language $\mathcal{L}$ whose sentences are built from the atomic sentences in the set $\mathrm{At} = \{p_0, p_1, \ldots\}$ using $\wedge, \vee, \neg, \Box$. A *Kripke model* M is a triple (W, R, V) where

- W is a set of worlds (so far, this was the set of prime filters $\mathsf{St}(L)$)
- $R \subseteq W \times W$ is a binary relation (so far, this was the one defined in 5.1)
- V is a function that assigns each possible world $x \in W$ to a function that assigns each atomic sentence a truth-value in $\{0, 1\}$, i.e., $V(x) : \mathrm{At} \to \{0, 1\}$ (so far, V was given by the map $F \mapsto \chi_F$ which maps an ultrafilter to its characteristic function, which in turn assigns truth

values not only to 'atomic' propositions but all propositions) this was given by the characteristic function χ_F of an ultrafilter)

and the valuation of atomic sentences is extended to all sentences: We recursively define when a sentence φ is true at a world x, written $x \vDash \varphi$:

- $x \vDash p$ iff $V(x)(p) = 1$
- $x \vDash \varphi \wedge \psi$ iff $x \vDash \varphi$ and $x \vDash \psi$
- $x \vDash \varphi \vee \psi$ iff $x \vDash \varphi$ or $x \vDash \psi$
- $x \vDash \neg\varphi$ iff $x \nvDash \varphi$
- $x \vDash \Box\varphi$ iff for all $y \in W$, if xRy, then $y \vDash \varphi$.

What the Kripke semantics omits is the topological structure on the set of possible worlds that we still have on our topological approach. This additional topological information amounts to the following: In the usual Stone duality, Boolean algebra homomorphisms correspond to continuous functions on the dual spaces. Now, we've generalized Boolean algebra homomorphisms to functions that preserve $\wedge$ and $\top$, and on the dual side they correspond to the relations defined in 5.1. The additional topological properties of these relations—i.e., the appropriate generalization of continuity—is that they are *Boolean compatible*: If X and Y are Stone spaces, a relation $R \subseteq X \times Y$ is Boolean compatible if (1) for all $x \in X$, the set $\{y : xRy\} \subseteq Y$ is closed and (2) for all clopen $U \subseteq Y$, the set $\{x : \exists y \in U.xRy\} \subseteq X$ is clopen. Then the functions $f : B \to A$ from a Boolean algebra B to a Boolean algebra A which preserve $\wedge$ and $\top$ are in one-to-one correspondence with the Boolean compatible relations $R \subseteq X \times Y$ on the dual spaces X of A and Y of B(Gehrke and van Gool 2023, cor. 4.43). If we have a Boolean space X with Boolean compatible relation R, we can find the dual Boolean algebra $B = \mathsf{Clp}(X)$ with $\Box U := \{x \in X : \forall x' \in X.xRx' \Rightarrow x' \in U\}$ (Gehrke and van Gool 2023, cor. 4.51).

Note the swap of direction, as usual for dualities

5.2 Exercises

Exercise 5.a. Tell the story that we've told in section 5.1 but now not for the 'propositions vs. possible worlds' duality, but for the 'properties vs. objects' duality. Here the new operator is Δ read as 'definitely' instead of $\Box$ read as 'necessarily'. This operator is used a lot in theories of vagueness (e.g. Williamson 1999). If p is the property of being red, then Δp is the property of definitely being red. Which semantic of 'definitely' do you get on the dual side? How does it compare to existing semantics? What does the topological perspective add? Is it philosophically plausible?

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